Week 4 Exercises - Selected Solutions

(1) (Angus) We will demonstrate the zero object, kernels and cokernels, and the isomorphism between co-image and image of a morphism.

First, to say a short exact sequence of functors

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is exact is to say

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$$

is exact in $\underline{\text{Ab}}$ for all $U \in \mathcal{C}$.

We note that the zero object is the functor $0(U) = 0$.

Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism.

We construct $\ker \alpha$ by setting $(\ker \alpha)(U)$ to be the object such that

$$0 \longrightarrow (\ker \alpha)(U) \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \mathcal{G}(U)$$

is exact. Cokernels similarly.

We have $\text{co-im}\, \alpha = \text{coker}(\ker \alpha \to \mathcal{F}$ and $\text{im}\, \alpha = \ker(\mathcal{G} \to \text{coker}\, \alpha)$. We get a morphism

$$\beta : \text{co-im}\, \alpha \longrightarrow \text{im}\, \alpha.$$

However, we see that for each $U \in \mathcal{C}$, we have

$$\beta(U) : (\text{co-im}\, \alpha)(U) = \text{co-im}(\alpha(U)) \longrightarrow (\text{im}\, \alpha)(U) = \text{im}(\alpha(U)),$$

which is an isomorphism since $\underline{\text{Ab}}$ is abelian. Thus by the exactness criterion above, β is an isomorphism, as required.

(2) (Angus) We wish to show $\mathcal{P}'$ is a sheaf. Thus, given an open U and an open cover $(U_i \to U)_{i \in I}$, we must show

$$\mathcal{P}'(U) \to \prod_{i \in I} \mathcal{P}'(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathcal{P}'(U_i \times_U U_j)$$

is an equalizer. We will denote the maps by restriction.

Assume we have $(s_i) \in \prod_{i \in I} \mathcal{P}'(U_i)$ such that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all $(i,j) \in I^2$. Since $\mathcal{F}$ is a sheaf, we have an element $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

By the definition of $\mathcal{P}'$, for each U_i we have a cover $(V_{i,k} \to U_i)_{k \in K}$ such that $s_i|_{V_{i,k}} \in \mathcal{P}(V_{i,k})$. By composition, we get a cover $(V_{i,k} \to U_i \to U)_{(i,k) \in I \times K}$ of U.

Thus we have an element $s \in \mathcal{F}(U)$ and a cover $(V_{i,k} \to U_i \to U)_{(i,k) \in I \times K}$ of U such that $s_i|_{V_{i,k}} \in \mathcal{P}(V_{i,k})$, so by the definition of $\mathcal{P}'$, we have $s \in \mathcal{P}'(U)$, as required.

(3) (Ricky) We will use Milne's definitions for exactness using Hom sets, i.e. $0 \to A \to B \to C$ is exact iff

$$0 \to \mathrm{Hom}(T, A) \to \mathrm{Hom}(T, B) \to \mathrm{Hom}(T, C)$$

is exact for all objects T.

(a) Let R be a functor admitting a left adjoint L, and suppose $0 \to A \to B \to C$ is exact. We want to show that $0 \to R(A) \to R(B) \to R(C)$ is exact. This is equivalent, by definition, to $0 \to \mathrm{Hom}(T, R(A)) \to \mathrm{Hom}(T, R(B)) \to \mathrm{Hom}(T, R(C))$ being exact. But this sequence is the same as

$$0 \to \mathrm{Hom}(L(T), A) \to \mathrm{Hom}(L(T), B) \to \mathrm{Hom}(L(T), C)$$

which is exact by definition of $0 \to A \to B \to C$ being exact. $\square$

(b) Follows immediately from (a) by duality (see (e)).

(c) Let R be a functor with an exact left adjoint L. Let I be an injective object; we show that $R(I)$ is as well. This means we must show that $\mathrm{Hom}(-, R(I))$ is an exact functor. Let $0 \to A \to B \to C \to 0$ be a short exact sequence. Then by assumption, $0 \to L(A) \to L(B) \to L(C) \to 0$ is also exact. As I is injective, we have

$$0 \to \mathrm{Hom}(L(C), I) \to \mathrm{Hom}(L(B), I) \to \mathrm{Hom}(L(A), I) \to 0$$

exact, i.e.

$$0 \to \mathrm{Hom}(C, R(I)) \to \mathrm{Hom}(B, R(I)) \to \mathrm{Hom}(A, R(I)) \to 0$$

is exact, so $R(I)$ is injective. $\square$

(d) ("Right Adjoints Preserve Limits") By the hint, we need only show that our functor R with left adjoint L preserves products and equalizers. Consider $P := \prod_{i \in I} P_i$ a product of objects P_i. We want to show that $R(P) =$

$\prod_{i\in I} R(P_i)$. Consider the functor of points of $R(P)$. For any test object T we have

$$\begin{aligned}\mathrm{Hom}(T, R(P)) &= \mathrm{Hom}(L(T), P)\\ &= \prod_i \mathrm{Hom}(L(T), P_i)\\ &= \prod_i \mathrm{Hom}(T, R(P_i))\\ &= \mathrm{Hom}(T, \prod_i R(P_i))\end{aligned}$$

where we use the universal mapping property of the product on the second line and again on the last line. Hence $R(P)$ and $\prod R(P_i)$ have the same functor of points, so by Yoneda they are the same.

The argument for preservation of equalizers is similar (ask if you'd like to see it). □

(e) The corresponding statements that follow by duality are: left adjoints are right exact, a functor that admits an exact right adjoint preserves projective objects, and a functor admitting a right adjoint preserves colimits.

(4) (Ricky) Let $F : \underline{Sets} \to \underline{Top}$ be defined by $F(S) = (S, P_S)$, where P_S is the power set of S (i.e. $F(S)$ is S with the discrete topology). We want to show that $\mathrm{Hom}(F(X), Y) = \mathrm{Hom}(X, i(Y))$ for a set X and topological space Y. This is clear from the fact that any morphism of sets out of a discrete topological space is automatically continuous, and every continuous map of topological spaces is determined by its map of underlying sets. In other words, given a map $F(X) \to Y$ we get the corresponding map on underlying sets $X \to i(Y)$, and given $X \to i(Y)$, it is automatically a continuous map for $F(X) \to Y$.

To see that a similar left adjoint cannot exist for the category of X-schemes, we will show the forgetful functor $i : \underline{Sch/X} \to \underline{Sets}$ does not preserve limits (see (3)). We will do the case $X = \mathrm{Spec}(\mathbb{Z})$ (e.g. $\underline{Sch/X} = \underline{Sch}$) and leave the general case as an easy tweak.

As taking fiber products of $\mathrm{Spec}(\mathbb{Z})$ with itself leaves it unchanged, we have $\prod \mathrm{Spec}(\mathbb{Z}) \cong \mathrm{Spec}(\mathbb{Z})$. But then if the product is uncountable, we have

$$i(\prod \mathrm{Spec}(\mathbb{Z})) \neq \prod i(\mathrm{Spec}(\mathbb{Z}))$$

as the left side is countable while the right side is uncountable.

(If you don't like the countability argument because you're worried about infinite tensor products (as I am now a bit unsure about), an easier version for $\mathbb{R}$-schemes goes like this: take the product of $\mathrm{Spec}(\mathbb{C})$ with itself (as an $\mathbb{R}$-scheme), which is $\mathrm{Spec}(\mathbb{C}^2)$. This has two points, but the product of two points in the category of sets is just another point. So the forgetful functor doesn't preserve products here either.) □

(5) (Angus)
(a) We in fact know that sheafification is right exact by Q3(b), since it admits a right adjoint in the forgetful inclusion $i : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{PreSh}(X_{\text{ét}})$. If we show it preserve monomorphisms, it will then be exact.

Consider a monomorphism $\alpha : \mathcal{P} \to \mathcal{F}$ of presheaves and the associated morphism $a\alpha : a\mathcal{P} \to a\mathcal{F}$ of sheaves.

To show this is a monomorphism is to show the morphism of stalks $a\alpha_x : a\mathcal{P}_x \to a\mathcal{F}_x$ is injective in $\underline{\mathrm{Ab}}$ for each $x \to X$. However, the stalks of the sheafification are isomorphic to the stalks of the original presheaf, so the above is equal to $\alpha_x : \mathcal{P}_x \to \mathcal{F}_x$, which is injective. Thus $a\alpha$ is a monomorphism, and thus sheafification is in fact exact.
(b)

Proof. First note that since $\mathcal{G}$ admits a right adjoint, it is in fact exact. Further, since $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathcal{C}}$, every object and morphism in $\mathcal{C}$ in the image of $\mathcal{G}$, so it is sufficient to work with objects of the form $\mathcal{G}(X)$ and morphisms $\mathcal{G}(\phi)$, for X an object in $\mathcal{D}$ and $\phi : A \to B$ a morphism in $\mathcal{D}$.

Thus by exactness of $\mathcal{G}$ we can construct $\ker \mathcal{G}(\phi) = \mathcal{G}(\ker \phi)$ and $\mathrm{coker}\, \mathcal{G}(\phi) = \mathcal{G}(\mathrm{coker}\, \phi)$.

Finally we have

$$\begin{aligned}
\text{co-im}\,\mathcal{G}(\phi) &\cong \operatorname{coker}(\ker \mathcal{G}(\phi) \to \mathcal{G}(A)) \\
&\cong \operatorname{coker}(\mathcal{G}(\ker \phi) \to \mathcal{G}(A)) \\
&\cong \operatorname{coker}(\mathcal{G}(\ker \phi \to A)) \\
&\cong \mathcal{G}(\operatorname{coker}(\ker \phi \to A)) \\
&\cong \mathcal{G}(\ker(B \to \operatorname{coker} \phi)) \\
&\cong \ker(\mathcal{G}(B \to \operatorname{coker} \phi)) \\
&\cong \ker(\mathcal{G}(B) \to \mathcal{G}(\operatorname{coker} \phi)) \\
&\cong \ker(\mathcal{G}(B) \to \operatorname{coker} \mathcal{G}(\phi)) \\
&\cong \operatorname{im} \mathcal{G}(\phi),
\end{aligned}$$

(using functoriality and exactness of $\mathcal{G}$, as well as the fact that $\mathcal{D}$ is abelian) as required.

$\square$

(c) We know $\mathrm{Sh}(X_{\text{ét}})$ is additive, and $\mathrm{PreSh}(X_{\text{ét}})$ is abelian. Further, the forgetful inclusion i and sheafification a are each additive functors. The inclusion is right adjoint to sheafification and sheafification is left exact.

Finally $a \circ i = \mathrm{id}_{\mathrm{Sh}(X_{\text{ét}})}$, since the sheafification of a sheaf is itself.

Thus the conditions of the lemma are satisfied, and $\mathrm{Sh}(X_{\text{ét}})$ is an abelian category.

(6) (Ricky) We check that the sequence is exact at stalks. Let $A = \mathcal{O}_{X,\overline{x}}$ be the strictly local ring of X at $\overline{x}$. Then for a sheaf arising from a group scheme $\mathcal{G}$ we have $\mathcal{G}_{\overline{x}} = \mathcal{G}(A)$. In other words, we need to verify that

$$0 \to \mu_n(A) \to A^\times \to A^\times \to 0$$

is exact for any strict henselian ring A when $\mathrm{char}(k) \nmid n$. This is clear everywhere except at the surjectivity of the nth power map. Given $a \in A^\times$, consider the polynomial $T^n - a \in A[T]$. Then since $A/\mathfrak{m}$ is separably closed and $(nT^{n-1}, T^n - a) = 1$ in $A/\mathfrak{m}[T]$, it splits over the residue field. But then the henselian property lifts a root to a root over A, i.e. a is an nth power. (Note: The fact that $\mathrm{char}(k) \nmid n$ was crucial so that we knew the derivative of $T^n - a$ didn't vanish so we could lift a root. This is only an issue in the étale topology which only allows unramified covers, but is not an issue for the flat topology, where the sequence is still exact even if $\mathrm{char}(k) \mid n$.) $\square$

(7) (Angus)
(a) Consider an exact sequence of presheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0,$$

so in particular

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$$

is exact for all $U \to Y$ etale (recall the exactness criterion as in Q1).

To show

$$0 \longrightarrow \pi_*\mathcal{F} \longrightarrow \pi_*\mathcal{G} \longrightarrow \pi_*\mathcal{H} \longrightarrow 0$$

is exact we need to show

$$0 \longrightarrow \pi_*\mathcal{F}(V) \longrightarrow \pi_*\mathcal{G}(V) \longrightarrow \pi_*\mathcal{H}(V) \longrightarrow 0$$

is exact for all $V \to X$ etale. However, this sequence just equals

$$0 \longrightarrow \mathcal{F}(V \times_X Y) \longrightarrow \mathcal{G}(V \times_X Y) \longrightarrow \mathcal{H}(V \times_X Y) \longrightarrow 0,$$

which is exact by assumption.
(b) $\mathrm{Sh}(X_{\text{ét}})$ is a full subcategory of $\mathrm{PreSh}(X_{\text{ét}})$, which maps into it via the left exact forgetful inclusion $i : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{PreSh}(X_{\text{ét}})$. So a sequence of sheaves of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}$$

is exact in $\mathrm{Sh}(X_{\text{ét}})$ if and only if

$$0 \longrightarrow i\mathcal{F} \longrightarrow i\mathcal{G} \longrightarrow i\mathcal{H}$$

is exact in $\mathrm{PreSh}(X_{\text{ét}})$.

So to show

$$0 \longrightarrow \pi_*\mathcal{F} \longrightarrow \pi_*\mathcal{G} \longrightarrow \pi_*\mathcal{H}$$

is exact in $\mathrm{Sh}(X_{\text{ét}})$ we consider

$$0 \longrightarrow i\pi_*\mathcal{F} \longrightarrow i\pi_*\mathcal{G} \longrightarrow i\pi_*\mathcal{H}$$

in $\mathrm{PreSh}(X_{\text{ét}})$. However, this just equals

$$0 \longrightarrow \pi_*i\mathcal{F} \longrightarrow \pi_*i\mathcal{G} \longrightarrow \pi_*i\mathcal{H},$$

which is exact by Q7(a).