Week 5 Exercises

The notation in the exercises comes from Milne's *Lectures on Etale Cohomology (v2.21)*..

1. Let $L = L_2 \circ L_1$ where L_1 and L_2 are both left exact functors from abelian categories with enough injectives. If L_1 preserves injectives and $(R^r L_1)(X) = 0$ for some object X, then $(R^r L)(X) = (R^r L_2)(L_1 X)$.

2. Prove Theorem 9.7 on Milne's *Lectures on Etale Cohomology (v2.21)* (i.e. the Excision theorem).

3. Prove that $C^\bullet(\mathcal{U}, \mathcal{P})$ is a complex.

4. Let $\mathcal{U}$ be the covering of X consisting of a single Galois covering $Y \to X$ with Galois group G. Let $\mathcal{P}$ be a presheaf on X carrying disjoint unions to products.
   (a) Show that the complex
   $$\mathcal{P}(X) \to \mathcal{P}(Y) \to \mathcal{P}(Y \times_X Y) \to \mathcal{P}(Y \times_X Y \times_X Y) \to \cdots$$
   is isomorphic to the complex of inhomogeneous cochains for G acting on $\mathcal{P}(Y)$ (see Milne's CFT p62). Each map in the complex is the alternating sum of the maps given by the various projection maps.
   (b) Deduce that
   $$\check{H}^r(\mathcal{U}, \mathcal{P}) = H^r(G; \mathcal{P}(Y)) \quad \text{(group cohomology)}.$$

5. Show that the given a refinement $\mathcal{V}$ of an etale covering $\mathcal{U}$ of X, the induced map on Čech cohomology groups
   $$\rho(\mathcal{V}, \mathcal{U}) : \check{H}^r(\mathcal{U}, \mathcal{P}) \to \check{H}^r(\mathcal{V}, \mathcal{P})$$
   is independent of all choices.

6. Prove that $\check{H}^r(X, \mathcal{I}) = 0$ for all $r > 0$, for all injective sheaves $\mathcal{I}$.

7. Give an example of a quasi-compact scheme X such that there exists a finite subset of X which is not contained in any open affine subset.

8. Compute the following cohomology groups:
   (a) $H^q((\mathbb{A}^1_{\mathbb{C}} - \{0\})_{\acute{e}t}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ if $q = 1$, or 0 if $q > 1$.
   (b) $H^q((\mathbb{A}^1_{\mathbb{C}})_{\acute{e}t}, \mathbb{Z}/n\mathbb{Z}) = 0$ for $q \geq 1$.
   (c) $H^2((\mathbb{P}^1_{\mathbb{C}})_{\acute{e}t}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. [Hint: use parts (a) and (b).]

9. Use the spectral sequence
   $$\check{H}^r(X_{\acute{e}t}, \mathcal{H}^s(\mathcal{F})) \Rightarrow H^{r+s}(X_{\acute{e}t}, \mathcal{F})$$
   to deduce

(a) $\check{H}^1(X_{ét}, \mathcal{F}) \simeq H^1(X_{ét}, \mathcal{F})$.

(b) There is an exact sequence

$$0 \to \check{H}^2(X_{ét}, \mathcal{F}) \to H^2(X_{ét}, \mathcal{F}) \to \check{H}^1(X_{ét}, \mathcal{H}^1(\mathcal{F})) \to, \check{H}^3(X_{ét}, \mathcal{F}) \to H^3(X_{ét}, \mathcal{F}).$$

10. Prove Theorem 10.8 (Mayer-Vietoris sequence) on Milne's *Lectures on Etale Cohomology (v2.21)*. [Hint: Use the spectral sequence $\check{H}^r(\mathcal{U}, \mathcal{H}^s(\mathcal{F})) \Rightarrow H^{r+s}(X_{ét}, \mathcal{F})$ for the étale covering $\mathcal{U} = (U_0 \to X, U_1 \to X)$.]

11. Solve exercise III.3.17 in Milne's *Etale Cohomology (1980)*.

12. (Ricky) Show the category of presheaves $\mathrm{PreSh}(X_{et})$ has enough injectives. (Does it follow from some formal nonsense or do you need to recreate the other proof?)