Week 5 Exercises - Selected Solutions

(1) Choose an injective resolution of X:

$$0 \to X \to I^0 \to I^1 \to I^2 \dots.$$

To compute the derived functors of L_1, we first apply L_1 to get

$$0 \to L_1(X) \to L_1(I^0) \to L_1(I^1) \to L_1(I^2) \dots$$

where the objects $L_1(I^j)$ are injective by hypothesis. If we apply L_2 to this, we get

$$0 \to L_2(L_1(X)) \to L_2(L_1(I^0)) \to L_2(L_1(I^1)) \to L_2(L_1(I^2)) \to \dots.$$

We want to say that the cohomology at the rth place computes $(R^r L)(X)$, so that $(R^r L)(X) = (R^r L_2)(L_1(X))$. But this will only compute the correct cohomology at the rth place if the previous sequence was exact at the rth place, i.e. if $R^r(L_1(X)) = 0$, which is our assumption. □

(10) (Following Milne, filling in some details.) Let $\mathcal{U} = (U_0 \to X, U_1 \to X)$ be an etale covering of X. Note for a presheaf $\mathcal{P}$ of abelian groups on X_{et}, we have the following exact sequence, by definition of the Čech cohomology groups:

$$0 \to \check{H}^0(\mathcal{U}, \mathcal{P}) \to \Gamma(U_0, \mathcal{P}) \oplus \Gamma(U_1, \mathcal{P}) \to \Gamma(U_0 \times_X U_1, \mathcal{P}) \to \check{H}^1(\mathcal{U}, \mathcal{P}).$$

Now let $\mathcal{H}^s(\mathcal{F})$ be the presheaf $U \mapsto H^s(U, \mathcal{F}|U)$. This sequence becomes $(*)$

$$0 \to \check{H}^0(\mathcal{U}, \mathcal{H}^s(\mathcal{F})) \to H^s(U_0, \mathcal{F}) \oplus H^s(U_1, \mathcal{F}) \to H^s(U_0 \times_X U_1, \mathcal{F}) \to \check{H}^1(\mathcal{U}, \mathcal{H}^s(\mathcal{F})).$$

Then the Grothendieck spectral sequence applied to $Sh(X_{et}) \to PSh(X_{et}) \to Ab$ (the composition of the forgetful functor and $\check{H}^0$) gives

$$\check{H}^r(\mathcal{U}, \mathcal{H}^s(\mathcal{F})) \implies H^{r+s}(X, \mathcal{F}).$$

As $\check{H}^r(\mathcal{U}, \mathcal{H}^s(\mathcal{F})) = 0$ for $r > 1$ (our cover only has two terms), the spectral sequence is supported on the first two vertical lines on the rs-plane (the E_2 page). Looking at the arrows, we see that the E_∞ page is the E_2 page, i.e. the sequence degenerates on page 2. To compute the $H^{r+s}(X, \mathcal{F})$ term, we use the fact that the diagonals on the E_∞ page give a filtration of it. After drawing the diagram, this gives us a short exact sequence

$$0 \to \check{H}^1(\mathcal{U}, \mathcal{H}^s(\mathcal{F})) \to H^{s+1}(X, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \mathcal{H}^s(\mathcal{F})) \to 0$$

(for $s \geq 0$) since the left term embeds in the H^{s+1} term with cokernel the right term (look at the relation between the E_∞ page and the actual cohomology if this doesn't make sense, remembering our E_∞ page has only two columns in it).

We use this to splice $(*)$ together for all s, giving the desired long exact sequence. □