Week 6 Exercises

1. Show that a short exact sequence $1 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 1$ gives rise to a long exact sequence of pointed sets

$$1 \to \mathcal{G}'(X) \to \mathcal{G}(X) \to \mathcal{G}''(X) \to \check{\mathrm{H}}^1(X, \mathcal{G}') \to \check{\mathrm{H}}^1(X, \mathcal{G}) \to \check{\mathrm{H}}^1(X, \mathcal{G}'').$$

2. Show that the map $\mathcal{S} \mapsto c(\mathcal{S})$ is well-defined from principal homogeneous $\mathcal{G}$-spaces to $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{G})$ for any open cover $\mathcal{U}$ splitting $\mathcal{S}$.

3. Let $\mathcal{G}$ be the constant sheaf for the finite group G. Show there is a bijection between Galois coverings of X with group G and principal homogeneous spaces for $\mathcal{G}$ on X.

4. Compute $H^1(X_{et}, \mathcal{G})$ for $\mathcal{G} = \mathbb{Z}/\ell\mathbb{Z}$ for the following spaces:
   - $X = \mathrm{Spec}(k)$ for k a field.
   - $X = A$ an abelian variety over a field.
   - $X = \mathrm{Spec}(\mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of integers in a number field.

5. Define $H^1(X_{et}, \mathbb{Z}_\ell)$ to be $\varprojlim H^1(X_{et}, \mathcal{G}_n)$ where $\mathcal{G}_n$ is the constant sheaf for the group $\mathbb{Z}/\ell^n\mathbb{Z}$. Compute $H^1(X_{et}, \mathbb{Z}_\ell)$ for the spaces in the last exercise when possible.

6. (a) Prove Hilbert's Theorem 90: $H^1(\mathrm{Spec}(K)_{et}, \mathbb{G}_m) = 0$ for K a field.

   (b) Show that the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{k^s})$ is injective, for k^s a separable closure of k, where X is a variety defined over k. (Hint: Use the Grothendieck spectral sequence for global sections and fixed points of the Galois action; then look at the Five Term exact sequence for general spectral sequences.)

   (c) Show the previous result still holds replacing k with $\overline{k}$. (Hint: Flat cohomology.)

7. (a) Let $\pi : Y \to X$ be a finite map (e.g. a closed immersion, or Y is the normalization of X). Prove that $H^r(X_{et}, \pi_*\mathcal{F}) \cong H^r(Y_{et}, \mathcal{F})$ for any $\mathcal{F} \in \mathrm{Sh}(Y_{et})$.

   (b) Let $\mathcal{F}$ be a quasicoherent sheaf on Y_{et} (cf Exercises 3). Suppose $\pi : Y \to X$ is affine, in the sense that there exists an affine étale cover $(U_i \to X)$ such that $(U_i \times_X Y \to Y)$ is an étale cover by affine schemes. Show $H^r(X_{et}, \pi_*\mathcal{F}) \cong H^r(Y_{et}, \mathcal{F})$.