Week 6 Exercises - Selected Solutions

(4) When $X = \mathrm{Spec}(k)$, we have $H^1(X_{et}, \mathbb{Z}/\ell\mathbb{Z}) = \hom(\pi_1(X), \mathbb{Z}/\ell\mathbb{Z}) = \hom(G_k, \mathbb{Z}/\ell\mathbb{Z})$. (Note this is the same as the usual Galois cohomology $H^1(G_k, \mathbb{Z}/\ell\mathbb{Z})$ since G_k acts trivially on $\mathbb{Z}/\ell\mathbb{Z}$.)

When $X = A$ an abelian variety, we know that $\pi_1(A) = T(A)$, the full Tate module, so $H^1(X_{et}, \mathbb{Z}/\ell\mathbb{Z}) = \hom(T(A), \mathbb{Z}/\ell\mathbb{Z}) = A[\ell]$ by topological considerations (the hom's are continuous).

When $X = \mathrm{Spec}(\mathcal{O}_K)$, we have $H^1(X_{et}, \mathbb{Z}/\ell\mathbb{Z}) = \hom(\pi_1(X), \mathbb{Z}/\ell\mathbb{Z}) = \hom(\mathrm{Cl}(K), \mathbb{Z}/\ell\mathbb{Z}) \cong \mathrm{Cl}(K)[\ell]$, the ℓ-torsion in the class group of K. This follows from the fact that $\pi_1(X)$ classifies unramified extensions of K, and that $\hom(G, \mathbb{Z}/\ell\mathbb{Z})$ is isomorphic to the ℓ-torsion in G for any abelian group G. $\square$

(5) Using computations similar to (4), we get $H^1(A_{et}, \mathbb{Z}_\ell) = \hom(T(A), \mathbb{Z}_\ell) \cong \hom(T_\ell(A), \mathbb{Z}_\ell) =: T_\ell(A)^\vee$ for A an abelian variety again for topological reasons. We have $H^1(k, \mathbb{Z}_\ell) = \hom(G_k, \mathbb{Z}_\ell)$, and $H^1(\mathcal{O}_K, \mathbb{Z}_\ell) = \lim \hom(\mathrm{Cl}(K), \mathbb{Z}/\ell^n\mathbb{Z}) \cong \lim \mathrm{Cl}(K)[\ell^n]$, which is the Sylow ℓ-subgroup of $\mathrm{Cl}(K)$. $\square$

(6) (a) One can use any classical proof using the comparison to Galois cohomology, but the easiest method using the tools available is to note that $H^1(\mathrm{Spec}(k), \mathbb{G}_m)$ classifies line bundles on $\mathrm{Spec}(k)$, all of which must be trivial as the underlying topological space is a single point.

(b) We use the Hochschild-Serre spectral sequence (the derivation from the Grothendieck spectral sequence can be read read on pg 96 of the notes, or elsewhere). Specifically, we have a spectral sequence

$$E_2^{p,q} = H^p(G_k, H^q(X_{k^s}, \mathbb{G}_m)) \implies H^{p+q}(X, \mathbb{G}_m)$$

where X_{k^s} denotes the base change of X to a separable closure. The five term exact sequence, which holds for any E_2 spectral sequence, is in this case:

$$0 \to E_2^{1,0} \to H^1(X, \mathbb{G}_m) \to E_2^{0,1} \to E_2^{2,0} \to H^2(X, \mathbb{G}_m).$$

We will compute each term individually.

- $E_2^{1,0} := H^1(G_k, H^0(X_{k^s}, \mathbb{G}_m)) = H^1(G_k, (k^s)^\times)$. (This is because X is projective, so $\mathcal{O}_X(X_{k^s}) = k^s$, and thus $\mathcal{O}_X^\times(X_k^s) = (k^s)^\times$.) But this cohomology group vanishes by Hilbert 90, i.e. $E_2^{1,0} = 0$.

- $H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X)$, by comparison with Cech cohomology.
- $E_2^{0,1} := H^0(G_k, H^1(X_{k^s}, \mathbb{G}_m)) = \mathrm{Pic}(X_{k^s})^{G_k}$, using the same comparison, and definition of Galois cohomology in degree 0.
- The last two are not needed, but for completion sake we note that $E_2^{2,0} := H^2(G_k, H^0(X_{k^s}, \mathbb{G}_m)) = H^2(G_k, (k^s)^\times) = Br(k)$ by similar reasoning as before, and a standard fact from Galois cohomology. Also, $H^2(X, \mathbb{G}_m)$ is often called the cohomological Brauer group of X by analogy; it need not equal $Br(X)$ (defined more naturally using division algebra ideas)! (But $Br(X) \hookrightarrow H^2(X, \mathbb{G}_m)$ I think.)

Thus the first part of our sequence reads

$$0 \to 0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_{k^s})^{G_k}$$

which gives us what we want. (Note: Not all line bundles on X_{k^s} which are Galois invariant need descend to a line bundle on X; the difference seems to be measured by the Brauer group of k!)

(c) (There should be a way to modify the above argument for flat cohomology, as we know $H^1(\mathbb{G}_m)$ is still the Picard group for this topology, but I don't remember how at the moment. The problem is we cannot use the Galois group if k is not perfect, so we need to use a composition of other functors, or another method entirely. Regardless, the result is still true.) $\square$

(7) (a) We use the Leray spectral sequence. In particular, we have

$$H^r(X_{et}, R^s\pi_*\mathcal{F}) \implies H^{r+s}(Y_{et}, \mathcal{F}).$$

Because π is finite, we know that $\pi_* : Sh(Y_{et}) \to Sh(X_{et})$ is exact, so $R^s\pi_*\mathcal{F} = 0$ for $s > 0$. This means our spectral sequence is all zero's except along the r-axis, i.e. it degenerates on page 2 with $H^r(X_{et}, R^0\pi_*\mathcal{F}) = H^{r+0}(Y_{et}, \mathcal{F})$, or $H^r(X_{et}, \pi_*\mathcal{F}) = H^r(Y_{et}, \mathcal{F})$. $\square$

(b) This is the same proof as for (a), noting that if π is affine, then the stalks of $R^s\pi_*\mathcal{F}$ are a limit over quasicoherent cohomology of affine schemes, which is zero by classical algebraic geometry. $\square$