Week 7 Exercises

1. Prove (or read) the Galois cohomological statements required in the computation of $H^r(X_{\text{ét}}, \mathbb{G}_m)$. That is, for X a complete nonsingular curve with function field K and x a closed point,

   (a) $H^r(\text{Spec}K_{\overline{x}}, \mathbb{G}_m) = 0$, for $r > 0$

   (b) $H^r(G_K, (K^{\text{sep}})^{\times}) = 0$, for $r > 0$

2. Let Y be a variety over a field k, let n be an integer such that $\text{char}(k) \nmid n$, let M be a rank 1 $\mathbb{Z}/n\mathbb{Z}$-module. Show that there is a *canonical* isomorphism

$$H^r(Y_{\text{ét}}, \underline{M})) \cong H^r(Y_{\text{ét}}, \underline{\mathbb{Z}/n\mathbb{Z}}) \otimes_{\mathbb{Z}/n\mathbb{Z}} M$$

   where $\underline{N}$ means the constant sheaf of the module N.

3. For a finite locally constant sheaf $\mathcal{F}$ with stalks annihilated by $n \in \mathbb{Z}$, let $\check{\mathcal{F}}(1)$ be the sheaf defined by

$$\check{\mathcal{F}}(1)(V) = \text{Hom}_V(\mathcal{F}|_V, \mu_N|_V).$$

   Show that if $\mathcal{G} = \check{\mathcal{F}}(1)$ then $\mathcal{F} = \check{\mathcal{G}}(1)$.

4. Let X be a complete nonsingular curve over an algebraically closed field k. The case $\mathcal{F}$ being the constant sheaf $\mathbb{Z}/n\mathbb{Z}$ and $r = 1$, Poincaré Duality gives a perfect pairing

$$H^1(X_{\text{ét}}, \underline{\mathbb{Z}/n\mathbb{Z}}) \times H^1(X_{\text{ét}}, \underline{\mathbb{Z}/n\mathbb{Z}}) \to H^2(X_{\text{ét}}, \underline{\mu_n}).$$

   Tensoring the groups with μ_n (and using Q2 above), we get

$$H^1(X_{\text{ét}}, \underline{\mu_n}) \times H^1(X_{\text{ét}}, \underline{\mu_n}) \to H^2(X_{\text{ét}}, \underline{\mu_n} \otimes \underline{\mu_n}) \cong \mu_n.$$

   Show that this coincides with the Weil pairing on the n-torsion of the Jacobian

$$\text{Jac}(X)[n] \times \text{Jac}(X)[n] \to \mu_n.$$

5. (Ricky) Show that $H^r(X, \oplus_i \mathcal{F}_i) = \oplus_i H^r(X, \mathcal{F}_i)$ for sheaves $\mathcal{F}_i$ on X.