Week 7 Exercises - Selected Solutions

(5) We will need to assume that our direct sum is over a finite set. (Is it even true otherwise?) This ensures that the direct sum presheaf is already a sheaf (isomorphic to the product). Let $0 \to \mathcal{F}_i \to I_i^\bullet$ be an injective resolution for $\mathcal{F}_i$. Then $H^r(X, \mathcal{F}_i) = H^r(\Gamma(I_i^\bullet))$ by definition. As direct sums preserve injective objects (proved at the end), we see that $\oplus_i I_i^\bullet$ is an injective resolution of $\oplus_i \mathcal{F}_i$, so $H^r(X, \oplus_i \mathcal{F}_i) = H^r(\Gamma(\oplus_i I_i^\bullet))$ by definition as well.

Our argument is to prove the following chain of equalities:

$$\begin{aligned} H^r(X, \oplus_i \mathcal{F}_i) &= H^r(\Gamma(\oplus I_i)^\bullet) \\ &= H^r(\oplus \Gamma(I_i)^\bullet) \\ &= \oplus H^r(\Gamma(I_i)^\bullet) \\ &= \oplus H^r(X, \mathcal{F}_i). \end{aligned}$$

The first and last equalities follow by the definitions as noted above.

The second equality comes from the assumption that our indexing set is finite, which implies $\Gamma(\oplus I_i) = \oplus \Gamma(I_i)$.

The third equality comes from observing if $A_i \to B_i \to C_i$ is a complex with $f_i : A_i \to B_i$ and $g_i : B_i \to C_i$, then $\ker(\oplus g_i)/im(\oplus f_i) \cong \oplus \ker(f_i)/im(g_i)$.

The last thing we need to prove is that a direct sum of injective objects is still injective. We use the criterion that A is injective iff $\mathrm{Hom}(-, A)$ is exact. Now suppose A_i are injective objects. Then $\mathrm{Hom}(-, \oplus A_i) \cong \oplus \mathrm{Hom}(-, A_i)$, and a direct sum of exact functors is exact. (Use the last relation to justify the third equality above.) $\square$