Week 8 Exercises

1. For $\pi : Y \to X$ a Galois covering with group G, prove there is a spectral sequence (Hochschild-Serre):

$$E_2^{r,s} : H^r(G, H^s(Y_{et}, \mathcal{F}|_Y)) \implies H^{r+s}(X_{et}, \mathcal{F}).$$

2. Finish the proof of (14.12), i.e. explain why $H^r(U, \mathcal{F}) = 0$ for $r \geq 2$.

3. In this example we investigate the cohomology of a hypersurface, using Poincaré duality. Let $X = V(f) \subset \mathbb{P}^{m+1}$ be a smooth hypersurface, so that $\dim(X) = m$. Let $U = \mathbb{P}^{m+1} \backslash X$.

   (a) Show there are isomorphisms $H^r(X, \Lambda) \to H^{r+2}(\mathbb{P}^{m+1}, \Lambda(1))$ for $r > m$, and a surjective such map for $r = m$. Deduce that

   $$H^m(X, \Lambda) \cong H^m(\mathbb{P}^{m+1}, \Lambda) \oplus H^m(X, \Lambda)'$$

   for $H^m(X, \Lambda)'$ the kernel of the above surjection.

   (b) Use Poincaré duality in the form $H^r_c(U, \Lambda) \times H^{2m-r}(U, \Lambda(m)) \to H^{2m}_c(U, \Lambda(m)) \cong \Lambda$ a perfect pairing to prove there is a decomposition of graded cohomology:

   $$H^*(X, \Lambda) \cong H^*(\mathbb{P}^{m+1}, \Lambda) \oplus H^m(X, \Lambda)'.$$

4. Show that $\mathcal{F}^! = \ker(\mathcal{F} \to j_*j^*\mathcal{F})$, so that it is a sheaf.

5. Finish the induction step for the proof of the purity theorem.