Week 8 Exercises - Selected Solutions

(3) (a) We have a smooth pair $(X, \mathbb{P}^{m+1})$ of codimension 1. By the Gysin sequence, we have

$$\cdots \to H^{r-2}(X, \Lambda(-1)) \to H^r(\mathbb{P}^{m+1}, \Lambda) \to H^r(U, \Lambda) \to H^{r-1}(X, \Lambda(-1)) \to \ldots$$

for all $r \geq 2$. As U is affine of dimension $m+1$, we have $H^r(U, \Lambda) = 0$ when $r > m+1$. This gives us the isomorphism in question from the long exact sequence, and a surjection when $r = m$. The statement for the degree m cohomology is just the usual decomposition for a morphism of vector spaces: $f : V \to W$ gives $V \cong \ker(f) \oplus W$. (Note: This holds for Λ p-torsion, so that these are actually vector spaces.)

(b) By Poincaré duality, we get that $H^{m+r}(X, \Lambda) \cong H^{m-r}(X, \Lambda(-r))$. For $r > 0$, we have $H^{m+r} \cong H^{m+r+2}(\mathbb{P}^{m+1}, \Lambda(1))$, so putting these all together along with the middle degree m isomorphism above gives the correct statement. $\square$

(4) Given $U = X \backslash Z$, $j : U \to X$, we have that $\mathcal{F}^!(V) = \ker(\mathcal{F}(V) \to \mathcal{F}(\varphi^{-1}(U))$ for $\varphi : V \to X$ an étale open of X by definition. But $j_* j^* \mathcal{F}(V) = j^* \mathcal{F}(j^{-1}(V \times_X U)) = \mathcal{F}(\varphi^{-1}(U))$ (the notation is a bit weird since the pushforward is basically defined so this will work.) $\square$