Week 9 Exercises - Selected Solutions

1. (a) Every base change of a finite morphism is finite, and therefore closed.

   (b) This is basically a rephrasing of the definition: an algebraic variety X over k is complete if, for every k-variety T, the projection map $X \times T \to T$ is closed. But the map $X \times T \to T$ is the base change to T of the map $X \to \mathrm{Spec}(k)$, so this definition is exactly saying that $X \to \mathrm{Spec}(k)$ is proper.

2. (This is an expanded exposition of Remark 17.9 in Milne's notes).

   Let $P = \mathrm{Spec}(\mathbb{Q})$ and $\mathcal{F} = \underline{\mathbb{Z}/2\mathbb{Z}}$. By example 11.3 and problem 1(a) from week 2,

$$H^1(P; \underline{\mathbb{Z}/2\mathbb{Z}}) \simeq \mathrm{Hom}_{\mathrm{cts}}(\pi_1^{\text{ét}}(P), \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Hom}_{\mathrm{cts}}(G_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Hom}_{\mathrm{cts}}(G_{\mathbb{Q}}, \mu_2).$$

   Now, the Kummer sequence

$$0 \to \mu_2(\mathbb{Q}^{\mathrm{sep}}) \to (\mathbb{Q}^{\mathrm{sep}})^* \xrightarrow{2} (\mathbb{Q}^{\mathrm{sep}}) \to 0$$

   gives the long exact sequence

$$0 \to \mu_2(\mathbb{Q}) \to \mathbb{Q}^* \xrightarrow{2} \mathbb{Q}^* \to H^1(G_{\mathbb{Q}}, \mu_2) \to H^1(G_{\mathbb{Q}}, (\mathbb{Q}^{\mathrm{sep}})^*) = 0,$$

   where the last group is 0 by Hilbert's Theorem 90. So we conclude that

$$H^1(G_{\mathbb{Q}}, \mu_2) \simeq \mathbb{Q}^*/(\mathbb{Q}^*)^2.$$

   Finally, since $\mu_2(\mathbb{Q}^{\mathrm{sep}})$ is contained in $\mathbb{Q}$, we conclude that

$$H^1(P; \underline{\mathbb{Z}/2\mathbb{Z}}) \simeq \mathrm{Hom}_{\mathrm{cts}}(G_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z}) \simeq H^1(G_{\mathbb{Q}}, \mu_2) \simeq \mathbb{Q}^*/(\mathbb{Q}^*)^2,$$

   so in particular it is an infinite group. Hence, we see that statement (a) in the corollary is false for k not separably closed.

   (b) Let $X = \mathbb{A}^1$ over $k = k^{\mathrm{sep}}$ and let $\mathcal{F}\underline{\mathbb{Z}/p\mathbb{Z}}$. The Artin-Schreier sequence (example 7.9(b)):

$$0 \to \underline{\mathbb{Z}/p\mathbb{Z}} \to \mathbb{G}_a \xrightarrow{t \mapsto t^p - t} \mathbb{G}_a \to 0$$

   gives rise to the exact sequence

$$k[x] \xrightarrow{t \mapsto t^p - t} k[x] \to H^1(\mathbb{A}^1, \underline{\mathbb{Z}/p\mathbb{Z}}) \to H^1(\mathbb{A}^1, \mathbb{G}_a) = 0,$$

   from which we conclude $H^1(\mathbb{A}^1, \underline{\mathbb{Z}/p\mathbb{Z}}) \simeq k[x]/(x^p - x)$.

   But then (b) is not satisfied, since given a separably closed extension k' of k, the inverse image of $\underline{\mathbb{Z}/p\mathbb{Z}}$ on $\mathbb{A}^1_{k'}$ is again $\underline{\mathbb{Z}/p\mathbb{Z}}$, so that

$$H^1(\mathbb{A}^1, \underline{\mathbb{Z}/p\mathbb{Z}}) \simeq k[x]/(x^p - x) \not\simeq k'[x]/(x^p - x) \simeq H^1(\mathbb{A}^1_{k'}, \underline{\mathbb{Z}/p\mathbb{Z}}).$$

   Hence, statement (b) is not true in general without the hypothesis that X is complete.

3. (This is adapted from the first subsection of Milne's notes, section 18).

   Let U be a nonsingular affine curve over $k = \bar{k}$ and let $\mathcal{F} = \underline{\mathbb{Z}/n\mathbb{Z}}$ for some n coprime to char(k). In this situation, we (at least) want all the cohomology groups with compact support to be finite; however, we will see that this is not true with the above definition.

   Since U is affine over an algebraically closed field, then the only complete subvarieties of U are the finite subvarieties, and for a finite subvariety Z of U,

$$H^r_Z(U, \mathcal{F}) = \bigoplus_{z \in Z} H^r_z(U, \mathcal{F})$$

   From Proposition 14.3, we know that for each (closed) point of U,

$$H^r_z(U, \mathcal{F}) = \begin{cases} \mathcal{F}(-1), & \text{if } r = 2, \\ 0, & \text{otherwise.} \end{cases}$$

   Hence applying the above definition we would get

$$H^r_z(U, \mathcal{F}) = \varinjlim_{Z} H_Z(U, \mathcal{F}) = \bigoplus_{z \in U} H^r_z(U, \mathcal{F}) = \begin{cases} \bigoplus_{z \in U} \mathcal{F}(-1), & \text{if } r = 2, \\ 0, & \text{otherwise.} \end{cases}$$

   In particular, for $r = 2$ the group obtained this way is not finite.