On the Cohomology of Moduli of Mixed Characteristic Shtukas

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Overview

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The Fargues-Fontaine Curve

- Fix: p prime, $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$.
- Recall the Fargues-Fontaine curve:

$$X_{\mathbb{C}_p} = Y_{\mathbb{C}_p}/\varphi^{\mathbb{Z}} := (\mathrm{Spa}(W(\mathcal{O}_{\mathbb{C}_p^\flat})) - \{|\cdot|_x : |p[\varpi^\flat]|_x = 0\})/\varphi^{\mathbb{Z}}.$$

- It is a "curve" in the sense that its local rings are DVRs.
- Also have a relative version $X_{(R,R^+)}$ for (R, R^+) a perfectoid pair, replacing $\mathcal{O}_{\mathbb{C}_p^\flat}$ above with $R^{+\flat}$ which glues to give X_S for S a perfectoid space.
- When S is in characteristic 0, X_S comes with a natural divisor.

Isocrystals

- Let $B(\mathrm{GL}_n) = \{g \in \mathrm{GL}_n(\breve{\mathbb{Q}}_p)\}/(g \sim bg\sigma(b)^{-1})$.
- We have a bijection $b \mapsto (\breve{\mathbb{Q}}_p^n, [\varphi] = b\sigma)$ from $B(\mathrm{GL}_n)$ to the set of isomorphism classes of $\overline{\mathbb{F}}_p$-isocrystals of rank n.
- By Dieudonne-Manin, the latter are classified by tuples of rational numbers (slopes). Say $b \in B(\mathrm{GL}_n)$ is basic if the corresponding isocrystal has only one slope, i.e. the Newton polygon is *isoclinic*.

Vector Bundles on the Curve

- When $S = \mathrm{Spa}(K, K^\circ)$ for K an algebraically closed perfectoid field, we can classify vector bundles on $X_K := X_S$ in terms of isocrystals.

Theorem (Fargues-Fontaine)

The category of vector bundles on X_K is semisimple. There is a faithful essentially surjective additive functor from the category of $\overline{\mathbb{F}}_p$-isocrystals to the category of vector bundles on X_K, sending $b \in B(\mathrm{GL}_n)$ to a rank n vector bundle denoted $\mathcal{E}_b$.

- When b is basic (corresponding to slope λ), we write $\mathcal{E}_b = \mathcal{O}(\lambda)$.

Example Bundles

- We have the trivial rank n vector bundle $\mathcal{O}^n$, which is isoclinic of slope 0, with automorphism group $\mathrm{GL}_n(\mathbb{Q}_p)$.
- Line bundles on X_K are of the form $\mathcal{O}(n)$ for $n \in \mathbb{Z}$, similar to $\mathbb{P}^1$. Their automorphism groups are $\mathrm{GL}_1(\mathbb{Q}_p)$.
- We have the simple rank 2 vector bundle (of "degree 1") given by $\mathcal{O}(1/2)$. It has automorphism group $\mathrm{Aut}(\mathcal{O}(1/2)) = D_{1/2}^\times(\mathbb{Q}_p)$.

Bundles on the Relative Curve

- For the relative curve, there is no map of adic spaces $X_S \to S$!
- However, for the purposes of doing sheaf theory, one can "pretend" there is: given $T \to S$, there is a map $X_T \to X_S$.
- For S a general perfectoid space, we have a bundle $\mathcal{E}_{X_S,b}$ determined by the condition that if $\mathrm{Spa}(K, K^\circ) \to S$ is a point of S with K complete & algebraically closed, then its pullback to X_K is $\mathcal{E}_{X_K,b}$.

Shtuka Data

- Let $G = \mathrm{GL}_n$, with its Borel of upper triangular matrices, and T the maximal torus of diagonal matrices.
- Let $\mu \in X_*(T)^+$ be a dominant positive cocharacter of GL_n, identified with a tuple of integers $(k_1, k_2, \ldots, k_n)$ where $k_1 \geq k_2 \geq \ldots k_n \geq 0$.
- For $\mu = (k_i), \nu = (\ell_i) \in X_*(T)^+$, write $\mu \geq \nu$ for the Bruhat ordering: $\sum_{i=1}^{j} k_i \geq \sum_{i=1}^{j} \ell_i$ for all j, with equality when $j = n$. (E.g. $(3,0,0) \geq (2,1,0) \geq (1,1,1)$.)
- For $b \in B(G)$, set $J_b = \{g \in G(\breve{\mathbb{Q}}_p) : gb\sigma(g)^{-1} = b\}$. When b is basic, this is an inner form of G.
- We will use the data (b, b', μ) with $b, b' \in B(G)$ and $\mu \in X_*(T)^+$ to define shtukas.

Mixed Characteristics Shtukas

- A shtuka (over K algebraically closed perfectoid) for (b, b', μ) is an injective map of vector bundles

$$\alpha : \mathcal{E}_b \to \mathcal{E}_{b'}$$

where $\mathrm{coker}(\alpha)$ is supported on the natural divisor of X_K (i.e. a fixed point x), and $\mathrm{coker}(\alpha_x)$ is a torsion $\mathcal{O}_{X,x}$-module of type μ, i.e. $\alpha_x : \mathcal{F}_x \to \mathcal{E}_x$ has elementary divisors compatible with μ.

- Recall: for example $\mathbb{Z}_p^2 \to \mathbb{Z}_p^2$ given by

$$\begin{bmatrix} p^3 & 0 \\ 0 & p \end{bmatrix}$$

has cokernel of type $\mu = (3, 1)$.

Moduli of Shtukas

- Given S a perfectoid space, a shtuka over S for (b, b', μ) is given by an injective map $\alpha : \mathcal{E}_{X_S,b} \to \mathcal{E}_{X_S,b'}$ as before using the corresponding bundles on the relative curve X_S.
- Write $\mathrm{Sht}_{(b,b',\mu)}(S)$ for the set of shtukas over S for (b, b', μ) and $\mathrm{Sht}_{(b,b',\leq\mu)}(S)$ for the shtukas over S of type $\leq \mu$.

Theorem (Scholze)

The pro-étale sheaf $S \mapsto \mathrm{Sht}_{(b,b',\mu)}(S)$ is a diamond.

- Note that $J_b(\mathbb{Q}_p)$ and $J_{b'}(\mathbb{Q}_p)$ act on this space via automorphisms of the domain/codomain of the α's. These actions commute, so the cohomology of $\mathrm{Sht}_{(b,b',\mu)}$ will be a representation for both of these groups, along with the Weil group.

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The IC_μ Sheaf

- The spaces $\mathrm{Sht}_{(b,b',\mu)}$ should be thought of as generally "not smooth" so their usual constant sheaf cohomology may be less well-behaved.
- Let Gr_G be the B_{dR} affine Grassmanian, parametrizing injective maps $\mathcal{E}_b \to \mathcal{E}$ with cokernel supported at x. We have a natural period map $\pi : \mathrm{Sht}_{(b,b',\mu)} \to \mathrm{Gr}_G$.
- For each $\mu \in X_*(T)$ there exists a "perverse sheaf" IC_μ on Gr_G. We also write IC_μ for the sheaf $\pi^* IC_\mu$ on $\mathrm{Sht}_{(b,b',\mu)}$.
- In the case that μ is minuscule, this is just a shifted constant sheaf.

The *IC* Cohomology

- Let

$$H(b, b', \mu) = \varinjlim_{K \subset J_b(\mathbb{Q}_p)} R\Gamma_c(\mathrm{Sht}_{(b,b',\leq\mu)}/K, IC_\mu).$$

- Given a representation π of J_b, write

$$H(b, b', \mu)[\pi] := RHom_{J_b}(H(b, b', \mu), \pi),$$

which lives in the derived category of representations of $J_{b'} \times W_{\mathbb{Q}_p}$.

The Lubin-Tate Tower

- Let π be a supercuspidal representation of GL_n and $b \in B(\mathrm{GL}_n)$ basic. Write π_b for the corresponding representation of J_b under Jacquet-Langlands and σ_π the representation of the Weil group under the Local Langlands correspondence.
- Let $\mu = (1, 0, \ldots, 0)$ and $b \in B(\mathrm{GL}_n)$ correspond to the simple isocrystal of slope $1/n$.
- A theorem of Scholze-Weinstein realizes $\mathrm{Sht}_{(I_n, b, \mu)}$ as the diamond associated to the Lubin-Tate tower for GL_n at infinite level.
- Hence

$$H(I_n, b, \mu)[\pi] \cong \pi_b \boxtimes \sigma_\pi.$$

Two Conjectures on Cohomology

- Some conjectures predict what the π-component in cohomology of Rapoport-Zink spaces (i.e. when μ is minuscule) should look like when π is a supercuspidal representation of J_b. We can generalize to the moduli of shtukas for μ not minuscule to get the following conjectures.
- Kottwitz: (K) If b and b' are basic, then

$$H(b, b', \mu)[\pi] \cong \pi_{b'} \boxtimes r_\mu \circ \sigma_\pi,$$

where r_μ is the irrep of GL_n of highest weight μ.

- Harris-Viehmann: (H-V) If b or b' is not basic, then

$$H(b, b', \mu)[\pi] = 0.$$

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Inductive Strategy

- Fix $G = \mathrm{GL}_n$. Consider the statements indexed by $\mu \in X_*(T)^+$:
- K(μ): (K) is true for (b, b', μ) for any $b, b' \in B(G)$.
- H-V(μ): (H-V) is true for (b, b', μ) for any $b, b' \in B(G)$.

Theorem (M.)

Suppose K(μ) and H-V(μ) hold for all μ minuscule. Then K(μ) holds for all $\mu \in X_*(T)^+$.

- Idea: Use twisting to go from K(μ) to K($\mu + (1, \ldots, 1)$) and a cohomological formula coming from geometry to spread out from those cases.

Motivating Calculation

- Let $G = \mathrm{GL}_2$. Consider injective maps $\alpha : \mathcal{O}^2 \to \mathcal{O}(1/2)$ and $\beta : \mathcal{O}(1/2) \to \mathcal{O}(1)^2$, each with cokernel supported at the natural divisor $x \in X$. Each are necessarily of type $\mu = (1, 0)$, so think of α as a point in the Lubin-Tate tower, and β like a point in a "dual tower."
- Composition gives a map $\beta \circ \alpha : \mathcal{O}^2 \to \mathcal{O}(1)^2$ with cokernel supported at x of type $(2, 0)$ or $(1, 1)$.
- Idea: Use Lubin-Tate spaces to compute cohomology of $\mathrm{Sht}_{(b,b',\leq(2,0))}$ where $\mathcal{E}_b \cong \mathcal{O}^2$ and $\mathcal{E}_{b'} \cong \mathcal{O}(1)^2$.

Motivating Calculation (2)

- Consider a pullback diagram of the form:

$$\begin{array}{ccc} \widetilde{\mathrm{Sht}} & \xrightarrow{\tilde{m}} & \mathrm{Sht}_{(b,b',\leq(2,0))} \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ \widetilde{\mathrm{Gr}}_{\leq(2,0)} & \xrightarrow{m} & \mathrm{Gr}_{\leq(2,0)} \end{array}$$

where $\widetilde{\mathrm{Gr}}_{\leq(2,0)}$ is the convolution Grassmanian parametrizing maps $\mathcal{O}^2 \to \mathcal{E} \to \mathcal{E}'$ of type $\leq (2,0)$.

- Then $\widetilde{\mathrm{Sht}}$ is the moduli space for pairs of maps $\alpha : \mathcal{O}^2 \to \mathcal{E}$ and $\beta : \mathcal{E} \to \mathcal{O}(1)^2$, with $\tilde{m}$ being $(\alpha, \beta) \mapsto \beta \circ \alpha$.

Motivating Calculation (3)

- We can stratify $\widetilde{\mathrm{Sht}}$ according to the possibilities for $\mathcal{E}$.
- The part where $\mathcal{E} \cong \mathcal{O}(1/2) := \mathcal{E}_{b_1}$ (the generic case) is

$$(\mathrm{Sht}_{(b,b_1,(1,0))} \times \mathrm{Sht}_{(b_1,b',(1,0))})/D_{1/2}^{\times}(\mathbb{Q}_p).$$

- The part where $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}(1) =: \mathcal{E}_{b_2}$ admits a similar presentation.

Motivating Calculation (4)

- Geometric Satake gives us that the cohomology of $\widetilde{\mathrm{Sht}}$ with coefficients in $IC_{(1,0)} \boxtimes IC_{(1,0)}$ equals the cohomology of $\mathrm{Sht}_{(b,b',\leq(2,0))}$ with coefficients in $IC_{(2,0)} \oplus IC_{(1,1)}$.
- Hence we get a distinguished triangle

$$\mathcal{B} \to \mathcal{A} \to \mathcal{C} \to^{+1}$$

where

$$\mathcal{A} = H(b, b', (2,0)) \oplus H(b, b', (1,1)),$$
$$\mathcal{B} = (H(b, b_1, (1,0)) \otimes H(b_1, b', (1,0))) \otimes_{D^\times_{1/2}(\mathbb{Q}_p)} \overline{\mathbb{Q}_\ell},$$
$$\mathcal{C} = (H(b, b_2, (1,0)) \otimes H(b_2, b', (1,0))) \otimes_{J_{b_2}(\mathbb{Q}_p)} \overline{\mathbb{Q}_\ell}$$

Motivating Calculation (5)

- Let π be a supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.
- By (H-V) for $\mu = (1,0)$, we get $\mathcal{C}[\pi] = 0$. Hence $\mathcal{A}[\pi] \cong \mathcal{B}[\pi]$.
- By (K) for $\mu = (1,0)$ and $(1,1)$, we get

$$H(b, b', (2,0)) \oplus \pi_{b'} \boxtimes \det \sigma_\pi \cong \pi_{b'} \boxtimes (\sigma_\pi \otimes \sigma_\pi).$$

- From $\sigma_\pi \otimes \sigma_\pi \cong \mathrm{Sym}^2 \sigma_\pi \oplus \det \sigma_\pi$, we deduce (K) for $\mu = (2,0)$, namely

$$H(b, b', (2,0))[\pi] \cong \pi_{b'} \boxtimes \mathrm{Sym}^2 \sigma_\pi.$$

Notation for General Formula

- For $\mu \in X_*(T)$, write V_μ for the corresponding irreducible representation of GL_n of highest weight μ.
- E.g. For $G = \mathrm{GL}_2, \mu = (1,0)$, we have $V_\mu = \mathrm{Std}$.
- Let $\mu_\bullet = (\mu_0, \mu_1, \dots, \mu_m)$ be a tuple of cocharacters of GL_n. Write $V_{\mu_\bullet}^\lambda$ for $Hom_{\mathrm{GL}_n}(V_\lambda, \otimes_i V_{\mu_i})$.
- E.g. for $G = \mathrm{GL}_2$ and $\mu_\bullet = ((1,0),(1,0))$, we have

$$\dim V_{\mu_\bullet}^\lambda = \begin{cases} 1 & \text{if } \lambda = (2,0) \text{ or } (1,1) \\ 0 & \text{else} \end{cases}$$

since $\mathrm{Std} \otimes \mathrm{Std} \cong \mathrm{Sym}^2(\mathrm{Std}) \oplus \det \mathrm{Std}$.

Geometric Satake Formula

Theorem (Imai)

Fix $b_0, b_m \in B(\mathrm{GL}_n)$. There is a distinguished triangle of the form

$$\mathcal{B} \to \mathcal{A} \to \mathcal{C} \to^{+1}$$

in the derived category of $J_b \times J_{b'} \times W_{\mathbb{Q}_p}$ representations, where

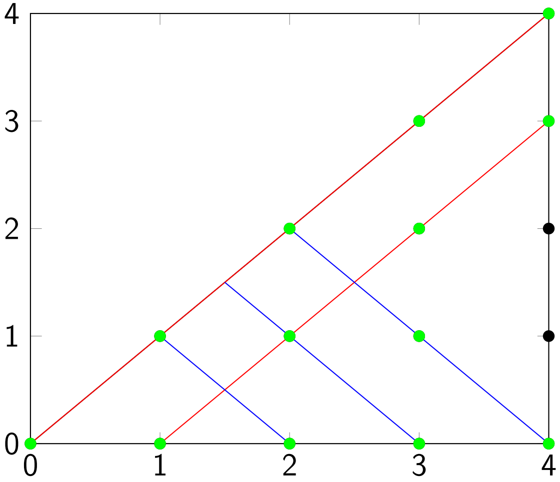
$$\mathcal{A} = \sum_{\lambda \in X_\bullet(T)^+} V_{\mu_\bullet}^{\lambda} \otimes H(b_0, b_m, \lambda),$$

$$\mathcal{B} = \bigotimes_{0 \le i \le m-1} H(b_i, b_{i+1}, \mu_i) \otimes_{\prod J_{b_i}} \overline{\mathbb{Q}_\ell},$$

and $\mathcal{C}$ coming from "non-basic" parts of the $\widetilde{\mathrm{Sht}}$ space.

The General GL_2 case

- Cocharacter lattice for GL_2. (Green = K(μ) true)



Picture for GL_n

- For general $G = \mathrm{GL}_n$, we have the *n*-dimensional lattice $X_*(T)$, and draw the positive dominant piece.
- The blue lines above become hyperplanes - the fibers of map taking $\mu = (k_i)$ to $\sum_i k_i$. These are linearly ordered.
- Minimal elements in the hyperplanes are twists of "smaller" cases, so can reduce the general case of $\mathrm{K}(\mu)$ to the minuscule case again.

Known Base Cases

- K(μ) is known for $G = \mathrm{GL}_2$ and GL_3 for μ minuscule by work of [Imai]. In fact, [Imai] deduces K(μ) for some non-minuscule cases for general $G = \mathrm{GL}_n$, which covers all K(μ) for $G = \mathrm{GL}_2$.
- (H-V) can be deduced for (b, b', μ) where the data (b, b', μ) is "Hodge-Newton reducible" from work of [Hansen]. Loosely this means the data arise from a proper Levi subgroup of G. This includes all of the relevant minuscule cases for GL_2, and many cases for GL_3. It excludes modifications of the form $\mathcal{O}(1/3) \to \mathcal{O}(1/2) \oplus \mathcal{O}(1)$ however.

Thanks for listening!