MA 341: Advanced Problems for Fun: # 4

Let \( f \) and \( g \) be arithmetic functions (maps from \( \mathbb{N} \) to \( \mathbb{C} \)). Define the convolution \( h \) of \( f \) and \( g \), written as \( f \ast g \), to be the function given by

\[
h(n) = (f \ast g)(n) := \sum_{d|n} f(d) \cdot g(n/d),
\]

where the sum is over the positive divisors \( d \) of \( n \).

This is an important operation on arithmetic functions we will explore a bit, and see their connection with \( L \)-functions.

1. Show that \( f \ast g = g \ast f \) and that \( (f \ast g) \ast h = f \ast (g \ast h) \). Also, \( f \ast (g + h) = f \ast g + f \ast h \). In other words, the set of arithmetic functions forms a ring under the usual + and multiplication given by \( \ast \).

2. Let \( I(n) = 1 \) if \( n = 1 \) and 0 otherwise. Show that \( f \ast I = f \) for any \( f \), so that \( I \) is the multiplicative identity in this ring.

3. Let \( 1(n) = 1 \) for all\( n \). Define \( \mu(n) \) via \( 1 \ast \mu = I \). Compute \( \mu(n) \) in terms of the prime factorization of \( n \).

4. (Mobius Inversion) Let \( g(n) = \sum_{d|n} f(d) \). Using the last problem, show that \( f(n) = \sum_{d|n} g(d)\mu(n/d) \). (i.e. convolution with \( \mu \) gets \( f \) out of the sum.)

5. Show that \( \varphi(n) = \sum_{d|n} d \cdot \mu(n/d) \), or equivalently \( n = \sum_{d|n} \varphi(d) \).

6. Show that \( \sum_{d|n} \tau(d) \cdot \mu(n/d) = 1 \) for all \( n \) if \( \tau(n) \) is the number of positive divisors of \( n \).

7. Let \( f \) be an arithmetic function. Define \( L(f, s) \) to be

\[
L(f, s) = \sum_{n \geq 1} \frac{f(n)}{n^s}.
\]

So for example, \( L(1, s) = \zeta(s) \), the usual Riemann zeta function, and \( L(f, s) = L(s) \) from class when \( f(n) = (-1)^{(n-1)/2} \). Show that

\[
L(f, s) \cdot L(g, s) = L(f \ast g, s)
\]

(this may remind you of a similar property for Fourier transform - this is no coincidence!). In particular, deduce that

\[
\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}.
\]

(Could you see this directly without using the identities above?)

8. Let \( \Lambda(n) \) equal either \( \ln(p) \) if \( n = p^k \) for \( p \) prime, or 0 otherwise. Show that \( \Lambda \ast 1 = \ln \).

9. Show that

\[
\frac{\zeta'(s)}{\zeta(s)} = -L(\Lambda, s),^1
\]

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^1 This is the first clue that the zeroes Riemann zeta function can be used to understand asymptotic behavior of \( \Lambda \) and hence be used to study the distribution of primes (though much more of an argument is needed to get to the Prime Number Theorem). The zeroes of \( \zeta(s) \) correspond to the “poles” of the above function, which place a major role in various integration techniques in complex analysis.