MA 341: Advanced Problems for Fun: # 4

Let f and g be arithmetic functions (maps from $\mathbb{N}$ to $\mathbb{C}$). Define the convolution h of f and g, written as $f * g$, to be the function given by

$$h(n) = (f * g)(n) := \sum_{d|n} f(d) \cdot g(n/d),$$

where the sum is over the positive divisors d of n.

This is an important operation on arithmetic functions we will explore a bit, and see their connection with L-functions.

1. Show that $f * g = g * f$ and that $(f * g) * h = f * (g * h)$. Also, $f * (g + h) = f * g + f * h$. In other words, the set of arithmetic functions forms a ring under the usual $+$ and multiplication given by $*$.

2. Let $I(n) = 1$ if $n = 1$ and 0 otherwise. Show that $f * I = f$ for any f, so that I is the multiplicative identity in this ring.

3. Let $\mathbf{1}(n) = 1$ for all n. Define $\mu(n)$ via $\mathbf{1} * \mu = I$. Compute $\mu(n)$ in terms of the prime factorization of n.

4. (Mobius Inversion) Let $g(n) = \sum_{d|n} f(d)$. Using the last problem, show that $f(n) = \sum_{d|n} g(d)\mu(n/d)$. (i.e. convolution with μ gets f out of the sum.)

5. Show that $\varphi(n) = \sum_{d|n} d \cdot \mu(n/d)$, or equivalently $n = \sum_{d|n} \varphi(d)$.

6. Show that $\sum_{d|n} \tau(d) \cdot \mu(n/d) = 1$ for all n if $\tau(n)$ is the number of positive divisors of n.

7. Let f be an arithmetic function. Define $L(f, s)$ to be

$$L(f, s) = \sum_{n \geq 1} \frac{f(n)}{n^s}.$$

So for example, $L(\mathbf{1}, s) = \zeta(s)$, the usual Riemann zeta function, and $L(f, s) = L(s)$ from class when $f(n) = (-1)^{\frac{n-1}{2}}$. Show that

$$L(f, s) \cdot L(g, s) = L(f * g, s)$$

(this may remind you of a similar property for Fourier transform - this is no coincidence!). In particular, deduce that

$$\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}.$$

(Could you see this directly without using the identities above?)

8. Let $\Lambda(n)$ equal either $\ln(p)$ if $n = p^k$ for p prime, or 0 otherwise. Show that $\Lambda * \mathbf{1} = \ln$.

9. Show that

$$\frac{\zeta'(s)}{\zeta(s)} = -L(\Lambda, s).$$ [1]

[1] This is the first clue that the zeroes Riemann zeta function can be used to understand asymptotic behavior of Λ and hence be used to study the distribution of primes (though much more of an argument is needed to get to the Prime Number Theorem). The zeroes of $\zeta(s)$ correspond to the "poles" of the above function, which place a major role in various integration techniques in complex analysis.