MA 341: Advanced Problems for Fun: # 1

Define the sequence of Fibonacci[1] numbers by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. These numbers arise in a few places in number theory, and have a lot of amazing properties. In doing these problems, it may help to write a long list of F_n's to be able to test some computations out.

1. (Relation to the Euclidean Algorithm) Let $S(a, b)$ be the number of steps it takes the Euclidean algorithm to terminate when computing the GCD of a and b. Show that if $a, b \leq F_n$, then $S(a, b) \leq S(F_n, F_{n-1})$. Show that $S(F_n, F_{n-1}) = n - 2$.

2. (Closed Form for F_n) Let

$$\varphi = \frac{1+\sqrt{5}}{2}, \text{ so } -\varphi^{-1} = \frac{1-\sqrt{5}}{2}.$$

   Show that

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}.$$

   Use linear algebra (eigenvalues/eigenvectors!) to show that

$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}.$$

3. (Speed of EA) Deduce from above that $n \approx \log(F_n)$, so that if $m = \max\{a, b\}$, then $S(a, b)$ is at most $\approx \log(m)$. (If you don't have perspective for speed of algorithms, this is incredibly fast!)

4. (Common Factors) Find an expression for (F_n, F_m). Use it to show that $(F_n, F_{n+1}) = 1$.

5. (Fun with 5) Let p be prime. Compare F_p with $\left(\frac{p}{5}\right)$ mod p. Can you prove your observation? (What's special about 5?)

6. (Convergents of φ) Compute the continued fraction for φ and some of the convergents. What do you notice? (If you can prove your observations, and believe convergents are good at approximating, then this gives one proof that $\lim_{n\to\infty} F_{n+1}/F_n = \varphi$. Can you prove it another way?)

[1] Once again, as is tradition in mis-naming objects, these numbers were at least studied quite a bit by Indian mathematicians going back to at least 200 BC, but Fibonacci (c. 1170 – c. 1250) gets the Westerners' acknowledgement for introducing the sequence there.