lecture notes for approximation area under graphs.

MA122, Lecture 7:

How do you really do integrals (when can't find antiderivative)?

Last time: For definite integrals.

\[ \int_a^b f(x) \, dx \]

Step 1: Choose \( \Delta x = \frac{b-a}{n} \), \( n = \# \text{of rectangles} \)
\( \Delta x = \text{width of each rectangle} \)

Step 2: Label points

\[ x_0 = a, \]
\[ x_1 = a + \Delta x, \]
\[ x_2 = a + 2\Delta x, \]
\[ \vdots \]
\[ x_n = a + n\Delta x = b. \]

Step 3: Find area of \( i \)th rectangle

(rectangle between \( x_i \) and \( x_{i+1} \))
Area (using "left hand rule")

\[ f(x_i) \Delta x \]

**Step 4. Add**

\[
\sum_{i=0}^{n-1} f(x_i) \Delta x + f(x_n) \Delta x + \ldots \\
= \sum_{i=0}^{n-1} f(x_i) \Delta x \\
\approx \int_a^b f(x) \, dx.
\]

**Question:** How close is the sum to the integral?

**What is the worst case error in each box?**
Note the faster the function increases or decreases,
the larger the error --

\[ K = \max_{a \leq x \leq b} |f'(x)| \]

Then max rate of increase (or decrease) is \( K \)
so worst case error

\[
\frac{\Delta y}{\Delta x} = \text{slope} = K \\
\text{in worst case}
\]

So \( \Delta y = K \Delta x \)

So worst case error (when \( f'(x) \)

is as large as possible, \( K \), for all \( x \)

we have area of error triangle

\[
\frac{1}{2} \Delta x \cdot \Delta y = \frac{1}{2} \Delta x (K \Delta x)
\]

\[
= \frac{1}{2} K \Delta x^2
\]

This is worst case for each rectangle

Total worst case error = worst case error

for each rectangle \(*\) rectangles
\[ \int_{a}^{b} f(x) \, dx = \frac{k \Delta x^2}{2} \cdot n \]

but \( n = \frac{b-a}{\Delta x} \)

So total worst case error

\[ = \frac{k \Delta x^2}{2} \cdot \frac{(b-a)}{\Delta x} \]

\[ = \frac{k}{2} (b-a) \Delta x \]

Good! Choose \( \Delta x \) small, worst case error is small. (We control choice of \( \Delta x \) -- "step!").

Example: (On a simple example)

\( f(x) = x^2 \)

\( a = 0, b = 2 \)

\[ \int_{0}^{2} x^2 \, dx \]

Pick \( \Delta x = 0.5 \).

Area under boxes is

\[ 0 \times 0.5 + (0.5)^2 \times 0.5 + 1 \times 0.5 \times (1.5)^2 \times 0.5 \]

\[ = 1.75 \]

What is worst case error.

\[ k = \max_{0 \leq x \leq 2} \left| \frac{d(x^2)}{dx} \right| = \max_{0 \leq x \leq 2} (2x) = 4 \]

\( b-a = 2-0 = 2 \)
so worst case is error with $\Delta x = 0.5$

$$\frac{4}{2} \cdot 2 \cdot (0.5) = 4 \times 0.5 = 2$$

So we can only guarantee that our approximation is within 2 of the actual area... (The actual area is)

$$\int_0^2 x^2 \, dx = \frac{x^3}{3} \bigg|_0^2 = \frac{8}{3} \approx 2.66$$

so our approximation $1.75$ is within 2.

Need much smaller $\Delta x$ to get better accuracy.

Or another idea -- what if we used trapezoids instead of rectangles

Area of $i$th trapezoid

$$\frac{f(x_i) + f(x_{i+1})}{2} \Delta x$$
So total approximate error is

\[ f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_n) \Delta x \]

or

\[ \frac{f(x_0) + f(x_n)}{2} \Delta x + \frac{f(x_1) + f(x_{n-1})}{2} \Delta x \]

\[ = \frac{1}{2} f(x_0) \Delta x + f(x_1) \Delta x + f(x_2) \Delta x + \cdots + f(x_{n-1}) \Delta x + \frac{1}{2} f(x_n) \Delta x. \]

We ask the same question about error -

If \( f(x) \) is a straight line there is no error, so the amount of error depends on the size of the 2nd derivative of \( f \).

The total error for the sum of trapezoids

\[ K_2 \cdot (b-a) \Delta x^2 \quad \text{-- This is better} \]

than rectangles because of the \( \Delta x^2 \) (small squared in very small).

For \( f(x) = x^2 \)

\[ \Delta x = 0.5 \]

\[ a = 0, b = 2 \]

The trapezoid area is 2.75

\[ \int_0^2 x^2 \, dx = \frac{x^3}{3} \bigg|_0^2 = 2.66 \ldots \]
Some integrals you should try on WolframAlpha.com:

\[
\text{Integrate } [\log(x), x] \\
\text{This is } \int \ln(x) \, dx \quad \text{"Log" means natural log to Wolfram.}
\]

\[
\text{Integrate } [x \times \log(x+4), x] \\
\int x \ln(x+4) \, dx
\]

\[
\text{Integrate } [x \times \sqrt{x+7}, x] \\
\int x \sqrt{x+7} \, dx
\]

\[
\text{Integrate } [(x^2) \times \sqrt{x+7}, x] \\
\int x^2 \sqrt{x+7} \, dx
\]

\[
\text{Integrate } [\exp(x+7), x] \quad \text{i.e. } \int e^{x+7} \, dx
\]

\[
\text{Integrate } [x^2 \exp(3x), \{x, 0, 5\}] \\
\int_0^5 x e^{3x} \, dx
\]