Homework for Oct 23 on power series and differential equations:

1. Find the 4th degree Taylor polynomial for the solution of

$$\frac{dy}{dx} = -3y + 2x^3$$

   with $y(0) = 5$.

2. Find the 5th degree Taylor polynomial for the solution of

$$\frac{dy}{dx} = x^2 y - \cos(x)$$

   (no initial condition is given, so $y(0) = a_0$.)

3. Find the 7th degree Taylor polynomial for the solution of

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

   (no initial conditions are given, so $y(0) = a_0$, $y'(0) = a_1$).

4. By collecting the terms that depend on $a_0$ and $a_1$ together, can you identify these power series (they are power series you have seen before!).

5. Find solutions of

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

   for $n = 0, 1, 2, 3$, that are polynomials (that is, for which the power series solution has only finitely many terms).

1. Given $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$
so $y'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$

and plugging in, we have

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots = -3(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) + 2x^2$$

So

$$\begin{aligned} a_1 &= -3a_0 \\ 2a_2 &= -3a_1 \\ 3a_3 &= -3a_2 + 2 \\ 4a_4 &= -3a_3 \\ &\vdots \end{aligned}$$

We are given $y(0) = 3$ so $a_0 = 3$

So $a_1 = -3(3) = -9.$

$2a_2 = -3a_1 \Rightarrow a_2 = \frac{-3\cdot(-9)}{2} = \frac{27}{2}$

$3a_3 = -3a_2 + 2 \Rightarrow a_3 = -a_2 + 2/3$

$a_3 = -\frac{27}{2} + \frac{2}{3} = -\frac{81}{6} + 4/6$

$a_3 = -77/6.$

$4a_4 = -3a_3 \Rightarrow a_4 = \frac{1}{4}(-3\cdot\frac{-77}{6}) = \frac{1}{4}\cdot(+\frac{77}{2})$

$a_4 = +\frac{77}{8}$

So $y(x) = 3 - 9x + \frac{27}{2}x^2 - \frac{77}{6}x^3 + \frac{77}{8}x^4$
is the 4th degree Taylor poly of the solution.

2. Again we guess

$$y(0) = a_0 + a_1 x + \dots + a_5 x^5 + \dots$$

so $$y'(0) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots$$

Plugging in, we have

$$a_1 + 2a_2 x + \dots + 5a_5 x^4 + \dots$$
$$= x^2 (a_0 + a_1 x + \dots + a_4 x^4 + \dots)$$
$$- \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)$$

So matching coefficients.

$a_1 = -1$ (constant coefficient

$2a_2 = 0$ ($x^1$ coefficient – no $x^1$ term on right)

$3a_3 = a_0 + \frac{1}{2}$ ($x^2$ coefficient)

$4a_4 = a_1 = -1$ ($x^3$ coefficient)

$5a_5 = a_2 - \frac{1}{24} = -1/24.$ ($x^4$ coefficient)

So $a_0$ is determined by $y(0) = a_0$ the initial condition

$a_1 = -1$

$a_2 = 0$

$a_3 = \frac{a_0}{3} + 1/6.$

$a_4 = -1/4$

$$a_5 = -1/24.$$

So the 5th degree Taylor polynomial of the solution is

$$y(x) = a_0 - x + (a_0 + 1/2)x^3 - x^4 - \tfrac{1}{24}x^5.$$

3. Ok looking for 7th degree Taylor poly.

Guess

$$y(x) = a_0 + a_1 x + \dots + a_7 x^7 + \dots$$

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots + 7a_7 x^6 + \dots$$

$$y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \dots + 42a_7 x^5 + \dots$$

$$= 2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + \dots + 7\cdot 6a_7 x^5 + \dots$$

Now plug in

$$2a_2 + 3\cdot 2a_3 x + \dots + 7\cdot 6a_7 x^5 + \dots$$
$$- 5(a_1 + 2a_2 x + 3a_3 x^2 + \dots + 6a_6 x^5 + \dots)$$
$$+ 6(a_0 + a_1 x + \dots + a_5 x^5 + \dots) = 0.$$

So equating coefficients

$$2a_2 - 5a_1 + 6a_0 = 0$$

$$3\cdot 2a_3 - 5\cdot 2a_2 + 6a_1 = 0$$

$$4\cdot 3a_4 - 5\cdot 3a_3 + 6a_2 = 0$$

$$\vdots$$

$$7\cdot 6a_7 - 5\cdot 6a_6 + 6a_5 = 0$$

So $a_2 = \frac{5a_1}{2\cdot 1} - \frac{6a_0}{2\cdot 1} = \frac{5}{2}a_1 - 3a_0$

$a_3 = \frac{5\cdot 2a_2}{3\cdot 2} - \frac{6a_1}{3\cdot 2}$

So $a_3 = \frac{5\cdot 2}{3\cdot 2}\left(\frac{5a_1}{2\cdot 1} - \frac{6a_0}{2\cdot 1}\right) - \frac{6a_1}{3\cdot 2}$

or $a_3 = \frac{5^2 a_1}{3\cdot 2\cdot 1} - \frac{5\cdot 6a_0}{3\cdot 1} - \frac{6a_1}{3\cdot 2} = \frac{25}{6}a_1 - 6a_0$

and $a_4 = \frac{5\cdot 3}{4\cdot 3}a_3 + \frac{6}{4\cdot 3}a_2$

$= \frac{5}{4}a_3 + \frac{1}{2}a_2$

and $a_4 = \frac{5}{4}\left(\frac{25}{6}a_1 - 6a_0\right) + \frac{1}{2}\left(\frac{5}{2}a_1 - 3a_0\right)$

$= \frac{125}{24}a_1 - \frac{15}{2}a_0 + \frac{5}{4}a_1 - \frac{3}{2}a_0$

$a_4 = \frac{155}{24}a_1 - \frac{18}{2}a_0 = \frac{155}{24}a_1 - 9a_0$

and $a_5 = \frac{5\cdot 4\, a_4}{5\cdot 4} - \frac{6a_3}{5\cdot 4} = a_4 - \frac{3}{10}a_3$

$a_5 = \frac{155}{24}a_1 - 9a_0 - \frac{3}{10}\left(\frac{25}{6}a_1 - 6a_0\right)$

also $a_6 = \frac{5\cdot 5a_5}{6\cdot 5} - \frac{6a_4}{6\cdot 5} = \frac{5}{6}a_5 - \frac{a_4}{5}$

$a_7 = \frac{5\cdot 6a_6}{7\cdot 6} - \frac{6a_5}{7\cdot 6} = \frac{5}{7}a_6 - \frac{a_5}{7}.$

4.) Answering this on "NO" is OK. -- however, they are "familiar".

We saw last week that guessing

$y = e^{rx}$ can give solutions

in this case (check this)

$y_1(x) = e^{3x}, \quad y_2(x) = e^{2x}$

We also checked that

$y(x) = ae^{3x} + be^{2x}$ is a solution

for any $a$ and $b$. (check this!)

So $y(0) = ae^{3 \cdot 0} + be^{2 \cdot 0} = a + b$

and $y'(0) = 3ae^{3 \cdot 0} + 2be^{2 \cdot 0} = 3a + 2b$.

So choosing $y(0) = a_0 = a + b$

$y'(0) = a_1 = 3a + 2b$

gives.

$$a + b = a_0$$

$$a = a_0 - b$$

so $3(a_0 - b) + 2b = a_1$

$$3a_0 - b = a_1$$

$$b = 3a_0 - a_1$$

$\Rightarrow \quad a_0 = a + 3a_0 - a_1$

so $a = a_1 - 2a_0$

Hence, $y(x) = (a_1 - 2a_0)e^{3x} + (3a_0 - a_1)e^{2x}$

or $y(x) = a_1(e^{3x} - e^{2x})$
$\qquad + a_0(3e^{2x} - 2e^{3x})$.

... not easy to guess...

5.) Again we guess power series

$$y(x) = a_0 + a_1 x + \dots + a_5 x^5 + \dots$$
$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots$$
$$y''(x) = 2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + 5\cdot 4a_5 x^3 + \dots$$

So
$$(1-x^2)(2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + 5\cdot 4a_5 x^3 + \dots)$$
$$-2x(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots)$$
$$+ n(n+1)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots) = 0$$

So

(constant term) $2a_2 + n(n+1)a_0 = 0$

($x^1$ term) $3\cdot 2a_3 - 2a_1 + n(n+1)a_1 = 0$

($x^2$ term) $4\cdot 3a_4 - 2a_2 - 4a_2 + (n(n+1))a_2 = 0$

($x^3$ term) $5\cdot 4a_5 - 3\cdot 2a_3 - 2\cdot 3a_3 + n(n+1)a_3 = 0$

$\vdots$

So
$$a_2 = \frac{-n(n+1)a_0}{2}$$

$$a_3 = \frac{2-n(n+1)}{3\cdot 2}a_1$$

$$a_4 = \frac{6a_2 - n(n+1)}{4\cdot 3}a_2$$

$$a_5 = \frac{+12a_3 - n(n+1)}{5\cdot 4}a_3$$

$\vdots$

If $a_0 = 0$ then all even #'d coefficients are 0

If $a_1 = 0$ then all odd #'d coefficients are 0.

Let $n=0$

then $a_2 = \frac{0}{2} \cdot a_0 = 0$

$a_3 = \frac{2}{3\cdot 2} a_1$

$a_4 = 0$ (because $a_2 = 0$)

$a_5 = \frac{12}{5\cdot 4} a_3$

$\vdots$

So if we let $a_0 = 1$, $a_1 = 0$ (so all odd #'d coefficients are zero)

then $y(x) = 1$ is a solution for $n=0$

Let $n=1$

$a_2 = \frac{-1\cdot 2}{2} a_0$

$a_3 = \frac{2-(1\cdot 2)}{3\cdot 2} a_1 = 0 \cdot a_1 = 0$

$a_4 = \frac{6a_2 - 1\cdot 2}{4\cdot 3} a_2$

$a_5 = 0$ because $a_3 = 0$.

So let $a_0 = 0$ (so all even #'d coefficients are 0) and $a_1 = 1$, then

$y(x) = 1 \cdot x$ is a solution for $n=1$.

Let $n=2$

$a_2 = \frac{-2\cdot 3}{2} a_0 = -3a_0$

$a_3 = \frac{2-2\cdot 3}{3\cdot 2} a_1$

$a_4 = \frac{6-(2\cdot 3)}{4\cdot 3} a_2 = 0 \cdot a_2 = 0$ (so all even #'d coeffs above $a_2$ are 0)

$a_5 = \frac{12-6}{5\cdot 4} a_3$

$\vdots$

So take $a_0 = 1$, $a_1 = 0$ (so all odd #'ed coeffs are 0)

$y(x) = 1 - 3x^2$ is a solution for $n = 2$

Let $n = 3$

$$a_2 = \frac{-3 \cdot 4}{2} a_0$$

$$a_3 = \frac{2 - (3 \cdot 4)}{3 \cdot 2} a_1 = \frac{2 - 12}{6} a_1 = -\frac{10}{6} a_1 = \frac{-5}{3} a_1$$

$$a_4 = \frac{6 - 3 \cdot 4}{4 \cdot 3} a_2$$

$$a_5 = \frac{12 - 3 \cdot 4}{5 \cdot 4} a_3 = 0 \cdot a_3 = 0$$ (so all higher odd #'d coeffs are also 0).

So take $a_0 = 0$ (so all the even #'d coeffs are 0)

$a_1 = 1$

$y(x) = x - \frac{5}{3} x^3$ is a solution for $n = 3$.