The process of creating new functions is one that you have seen many times over your career in mathematics. First linear functions, then quadratics, then rational functions, trig functions, exponenetials, and finally, logs.

For each of these types functions, we found connections to the previously defined functions. We get to know these functions and they start to seem natural–how else could they be other thant the way they are? They seem so natural, that we forget that they were invented. But it is interesting to look at the process of creating, then studying, then understanding a new type of function.

For example, you came across the natural log function as the antiderivative of $1/x$, that is, you *defined* natural log as

$$\ln(x) = \int \frac{1}{x}\, dx$$

where the constant of integration is chosen to so that $\ln(1) = 0$. Then you studied this function and proved interesting things about it. For example

$$\frac{d(\ln(e^x))}{dx} = \frac{1}{e^x} e^x = 1$$

so $\ln(x)$ and $e^x$ are inverse functions of each other. "But, of course", you say...because you have already see this before, it seems obvious that $\ln(x)$ is the inverse function of $e^x$.

For the trigonometric functions, the process is even more embedded. We defined the functions sine and cosine by looking at right triangle, Properties like $\sin^2 x + \cos^2 x = 1$ come from geometry. If you look back at your Calculus 1 book (or in Stewart) you will see that you used geometry to verify that

$$\frac{d(\sin(x))}{dx} = \cos(x) \text{ and } \frac{d(\cos(x))}{dx} = -\sin(x).$$

Hence, sine and cosine are solutions of the initial value problems

$$\frac{d^2y}{dx^2} = -y, \quad y(0) = 0, y'(0) = 1$$

and

$$\frac{d^2y}{dx^2} = -y, \quad y(0) = 1, y'(0) = 0,$$

respectively. We could also just start with saying these initial value problems are the definitions of sine and cosine, then derive the geomentric properties.

To illustrate these ideas better, let's define some new functions. These functions do come up in applied mathematics (see below).

Our first new function $y_1(x)$,is the solution of the initial value problem

$$\frac{d^2y}{dx^2} = y, \quad y(0) = 0, y'(0) = 1,$$

and the second, $y_2(x)$, is the solution of

$$\frac{d^2y}{dx^2} = y, \quad y(0) = 1, y'(0) = 0.$$

Because it gets annoying to refer to these as $y_1(x)$ and $y_2(x)$, we give them their standard names. The function $y_1(x)$ is called $\sinh(x)$ while the function $y_2(x)$ is called $\cosh(x)$ (these are pronounced "sinch" and "cosh").

OK, so what can se say about these functions? What interesting (perhaps useful) properties to do they have. All we know about them so far is their definitions above. That is, all we know is

$$\frac{d^2 \sinh}{dx^2} = \sinh(x), \quad \sinh(0) = 0, \sinh'(0) = 1,$$

and

$$\frac{d^2\cosh(x)}{dx^2} = \cosh(x), \quad \cosh(0) = 0, \cosh'(0) = 1.$$

Let's see if we can compute

$$\frac{d(\sinh(x))}{dx}.$$

To see what function this is, we notice that

$$\frac{d^2}{dx^2}\left(\frac{d(\sinh(x))}{dx}\right) = \frac{d^3(\sinh(x))}{dx^3} = \frac{d}{dx}\left(\frac{d^2(\sinh(x)}{dx^2}\right) = \frac{d(\sinh(x))}{dx},$$

where the last equality is from the fact that the second derivative of sinh is itself (the defining equation!). Hence

$$\frac{d^2}{dx^2}\left(\frac{d(\sinh(x))}{dx}\right) = \frac{d(\sinh(x)}{dx},$$

or the derivative of sinh is a solution of the differential equaiton

$$\frac{d^2y}{dx^2} = y.$$

Moreover, $(\sinh)'(0) = 1$, again by definition of sinh. Finally, $(\sinh)''(x) = \sinh(x)$ (again, by definition), so $(\sinh)''(0) = \sinh(0) = 0$.

What this all shows is that the derivative of sinh, $\sinh'(x)$ has second derivative equal to itself and has value 1 at $x = 0$ and derivative 0 at $x = 1$. But this is exactly the defining equation of cosh! Hence

$$\frac{d(\sinh)}{dx} = \sinh'(x) = \cosh(x).$$

Similarly, we can show that

$$\frac{d(\cosh)}{dx} = \cosh'(x) = \sinh(x).$$

This is reminiscent of sine and cosine–so let's try another trig identity for cosh and sinh...For example, is

$$\sinh^2(x) + \cosh^2(x) = 1?$$

If we differentiate

$$\frac{d(\sinh^2(x) + \cosh^2(x))}{dx} = 2\sinh(x)\cosh(x) + 2\cosh(x)\sinh(x)$$

which is not zero (try $x = 0$). Too bad...however, if we try a slightly different version...
FACT:

$$\cosh^2(x) - \sinh^2(x) = 1$$

This follows by noting that

$$\cosh^2(0) - \sinh^2(0) = 1$$

and

$$\frac{d(\sinh^2(x) - \cosh^2(x))}{dx} = 2\sinh(x)\cosh(x) - 2\cosh(x)\sinh(x) = 0$$

so $\cosh^2(x) - \sinh^2(x)$ is constant.

Now this really does start to look like trig functions. In fact, cosh and sinh are called "Hyperbolic Trig Functions". We can keep going and define hyperbolic cosine as $\frac{1}{\cosh(x)}$ and so on.

These functions are not just toys. It turns out that "hanging chains" (like power lines between two poles) are the shape of the graph of the $\cosh(x)$ function.

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