The following are due Weds, Oct 21. Focus on writing the solutions concisely, completely and neatly. Have someone else in the class check your work. Please follow the usual rules for homework, (see instructions on web page—e.g., 8.5x11 paper, stapled in the upper left corner. NOT crowded, etc.). Neatness counts!

1. For the sequence $\{a_n\}$, describe in a sentence the difference between
$$\lim_{n\to\infty} a_n \text{ and } \sum_{n=0}^{\infty} a_n$$

2. Compute (if you can)
$$\lim_{n\to\infty} \frac{n\sin(n)}{n^2+1} \text{ and } \sum_{n=0}^{\infty} \frac{n\sin(n)}{n^2+1}.$$

3. Determine if
$$\sum_{n=1}^{\infty} \frac{\cos(3n)}{1+(1.5)^n}$$
converges or diverges.

4. Compute
$$\sum_{n=1}^{\infty} \frac{3^{n+2}}{7^n}$$

5. Compute the radius of convergence of
$$\sum_{n=0}^{\infty} \frac{3^n(x+2)^n}{n^3}$$

6. Compute the fourth degree Taylor polynomial of $\ln(x)$ centered at $x = 3$.

7. Determine the maximum error between the fourth degree Taylor polynomial of $\ln(x)$ at $x = 3$ and $\ln(x)$ for $3 \le x \le 4$.

8. What is the Taylor series centered at zero for $e^{-x^2}$?

9. What is the Taylor series centered at zero for the function
$$f(x) = \int_0^x e^{-s^2}\, dx.$$

10. Find the tangent line to $y = x^2 - 3$ at $x = 1$ (This is a Calc 1 problem, but you now have a really quick way to do it!).

11. What is the maximum error between the tangent line and the function in the proceeding problem for $0.9 \le x \le 1.1$. Then repeat the problem for $0.99 \le x \le 1.01$.

12. Explain the following, commonly used statement:

    "The difference between a tangent line to a function and the function is quadratic in the distance from the point of tangency."

Solutions:

1. Well,

$\lim_{n\to\infty} a_n$ is the limit of the sequence $a_1, a_2, \ldots$

So it is the number $L$ such that as $n$ gets larger, $a_n$ gets closer to $L$ (if such a number exists),

while

$\sum_{n=1}^{\infty} a_n$ is the limit of the sequence of partial sums, i.e.

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

so it is the limit of the sequence

$$a_1,\ a_1+a_2,\ a_1+a_2+a_3,\ a_1+a_2+a_3+a_4,\ \ldots$$

2.) Well

$$\lim_{n\to\infty} \frac{n\sin(n)}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2(1+1/n^2)}\sin(n)$$

$$= \lim_{n\to\infty} \frac{1}{n(1+1/n^2)}\sin(n)$$

and we know $1+\frac{1}{n^2}\to 1$ and $|\sin(n)|\leq 1$ for all $n$

so $\lim_{n\to\infty} \frac{n\sin(n)}{n^2+1} = 0$

Also

$$\sum_{n=0}^{\infty} \frac{n\sin(n)}{n^2+1}$$ turns out to be a

<u>very</u> difficult problem (hence the "if you can")

3.) Well

$$1+(1.5)^n > (1.5)^n$$

So

$$\frac{1}{1+(1.5)^n} < \frac{1}{(1.5)^n}$$

So

$$\left|\frac{\cos(3n)}{1+(1.5)^n}\right| < \frac{1}{(1.5)^n} \text{ for all } n.$$

Since $\sum_{n=1}^{\infty} \frac{1}{(1.5)^n}$ converges (geometric series – equals $1.5 \cdot \frac{1}{1-1.5}$)

by the comparison test

$$\sum_{n=1}^{\infty} \left|\frac{\cos(3n)}{1+1.5^{-n}}\right| \text{ converges}$$

So $$\sum_{n=1}^{\infty} \frac{\cos(3n)}{1+1.5^{-n}} \text{ converges.}$$

4. Well

$$\sum_{n=1}^{\infty} \frac{3^{n+2}}{7^n} = \sum_{n=1}^{\infty} 9\cdot\left(\frac{3}{7}\right)^n = 9\cdot\frac{3}{7}\cdot\sum_{n=0}^{\infty}\left(\frac{3}{7}\right)^n$$

$$= 9\cdot\frac{3}{7}\cdot\frac{1}{1-3/7} = \frac{27}{7}\cdot\frac{1}{\left(\frac{4}{7}\right)} = \frac{27}{7}\cdot\frac{7}{4} = \frac{27}{4}.$$

5.) Using the ratio test

$$\lim_{n\to\infty}\left|\frac{3^{n+1}(x+2)^{n+1}}{(n+1)^3}\Big/\frac{3^n(x+2)^n}{n^3}\right|$$

$$=\lim_{n\to\infty}|3(x+2)|\cdot\frac{n^3}{(n+1)^3}$$

$$=|3(x+2)|.$$

So for convergence we need $3|x+2|<1$
or $|x+2|<1/3$.

Hence the radius of convergence is $1/3$.

(and the interval of convergence is centered at $x=-2$).

6.) Well

$\ln(3)=\ln(3)$

$\frac{d}{dx}\ln(x)=\frac{1}{x}$ so $\frac{d}{dx}\ln(x)\Big|_{x=3}=\frac{1}{3}$

$\frac{d^2}{dx^2}\ln(x)=-\frac{1}{x^2}$ so $\frac{d^2}{dx^2}\ln(x)\Big|_{x=3}=-\frac{1}{9}$

$\frac{d^3}{dx^3}\ln(x)=\frac{2}{x^3}$ so $\frac{d^3}{dx^3}\ln(x)\Big|_{x=3}=\frac{2}{27}$

$\frac{d^4}{dx^4}\ln(x)=\frac{-6}{x^4}$ so $\frac{d^4}{dx^4}\ln(x)\Big|_{x=3}=\frac{-6}{81}=-\frac{2}{27}$

So the Taylor poly. for $\ln(x)$ centered at $x=3$ of degree 4 is

$$T(x)=\ln(3)+\frac{1}{3}(x-3)-\frac{1/9}{2!}(x-3)^2+\frac{2/27}{3!}(x-3)^3-\frac{2/27}{4!}(x-3)^4$$

7. The Max error between $T(x)$ from #6 and $\ln(x)$ is

$$\left| \ln(x) - T(x) \right| \leq \frac{M_5}{5!} \left| x-3 \right|^5$$

Where $M_5 = \max \left| \frac{d^5 \ln(x)}{dx^5} \right|$

~~[illegible]~~ Now $\frac{d^5 \ln}{dx^5} = \frac{24}{x^5}$

and ~~[illegible]~~ on the interval $3 \leq x \leq 4$ this has its max value at $x=3$, so we may take

$$M_5 = \frac{24}{3^5}.$$

So for $3 \leq x \leq 4$

$$\left| \ln(x) - T(x) \right| \leq \frac{24/3^5}{5!} \left| x-3 \right|^5$$

Since $|x-3| \leq 1$ on this interval we ~~[illegible]~~ ~~[illegible]~~ also get

$$\left| \ln(x) - T(x) \right| < \frac{24}{3^5 \cdot 5!}$$

8) The Taylor series at $x=0$ for $e^x$ is $1 + x + \frac{x^2}{2!} + \cdots$

So $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$

9) Using #8

$$f(x) = \int_0^x e^{-s^2}\,ds$$

$$= \int_0^x 1 - s^2 + \frac{s^4}{2!} - \frac{s^6}{3!} + \frac{s^8}{4!} - \cdots\, ds$$

$$= s - \frac{s^3}{3} + \frac{s^5}{2!\cdot 5} - \frac{s^7}{3!\cdot 7} + \frac{s^9}{4!\cdot 9} - \cdots \Big|_0^x$$

$$= x - \frac{x^3}{3} + \frac{x^5}{2!\cdot 5} - \frac{x^7}{3!\cdot 7} + \frac{x^9}{4!\cdot 9} - \cdots$$

10. The tangent line is the first degree Taylor polynomial, Using

$$f(x) = x^2 - 3$$

so $f(1) = 1 - 3 = -2$

and $f'(x) = 2x$

so $f'(1) = 2$

So the tangent line is $T(x) = -2 + 2(x-1)$.

$$= -2 + 2x - 2$$

$$= 2x - 4.$$

11. Using Taylor's inequality

$$|(x^2-3) - T(x)| \le \frac{M_2}{2!}|x-1|^2$$

Now $M_2 \ge \max\left|\frac{d^2}{dx^2}(x^2-3)\right|$ so we may take $M_2 = 2$.

So $|(x^2-3) - T(x)| \le \frac{2}{2!}|x-1|^2$.

So for $.9 < x < 1.1$ we have

$$|(x^2-3) - T(x)| < .1^2 = 0.01$$

while for $.99 < x < 1.01$ we have

$$|(x^2-3) - T(x)| < .01^2 = 0.0001.$$

12.) The difference between a tangent line and and the function is less than or equal to a constant times the square of the distance to the point of tangency.