One More Money Problems

Suppose Ant decides they want to increase what is available in 40 years. They decide that they will increase their investment by 100 dollars every year by saving about 30 cents more each day. So, by the end of the first year they will be saving at a rate of 1100 dollars per year and at the end of the second year they will be saving at a rate of 1200 per year, and so on. Suppose they are still saving in a good investment, getting 5 percent per year interest compounded continuously.

How much money will Ant have at the end of 40 years?

Solution

First we must build a new model. We let \( M(t) \) be money saved at time \( t \) (in dollars and years, respectively) as before. Then the new model is

\[
\frac{dM}{dt} = 0.05M + (1000 + 100t), \quad M(0) = 0.
\]

Note that the left hand side of the differential equation is a rate of change, so has units Money/time (dollars/year), so the right hand side must have the same units. The term \( 0.05M \) comes from the interest rate and gives what the interest adds per year (so is in dollars per year). The \( 1000 + 100t \) is also in dollars per year because it is the RATE at which money is saved at time \( t \).

To solve this equation we use that it is linear. The natural response is

\[ M_g(t) = ke^{0.05t} \]

as before.

For the forced response, we guess

\[ M_p(t) = \alpha + \beta t \]

(we will see that we need both terms since the derivative of \( \beta t \) is just a constant). Plugging this into

\[
\frac{dM_p}{dt} = 0.05M_p + (1000 + 100t)
\]

we get

\[
\frac{d(\alpha + \beta t)}{dt} = 0.05(\alpha + \beta t) + 1000 + 100t
\]

or

\[ \beta = 0.05\alpha + 0.05\beta t + 1000 + 100t. \]

or

\[ \beta = 0.05\alpha + 1000 + (0.05\beta + 100)t. \]

In order for this equation to hold for all \( t \), we must have

\[ \beta = 0.05\alpha + 1000 \]

and

\[ 0 = 0.05\beta + 100. \]
The second equation gives
\[ \beta = \frac{-100}{0.05} = -2000 \]
and plugging this into the first equation gives
\[ -2000 = 0.05\alpha + 1000 \]
or
\[ -3000 = 0.05\alpha \]
or
\[ \alpha = \frac{-3000}{0.05} = -60000. \]
Hence, the forced response is
\[ M_p(t) = -60000 - 3000t. \]
The general solution is
\[ M(t) = ke^{0.05t} - 60000 - 3000t \]
and using \( M(0) = 0 \) we get the particular solution
\[ M(t) = 60000e^{0.05t} - 60000 - 3000t. \]

Plugging in \( t = 40 \) we get
\[ M(40) \approx 263343. \]
Note that the total investment by Ant would be
\[ \int_{0}^{40} 1000 + 100t \, dt = 1000t + \frac{100t^2}{2} \bigg|_{0}^{40} = 120000, \]
and 5000 per year would be the rate of investment at the end of the 40 years.
If Ant invested at a rate of 2500 dollars per year at 5 percent compounded continuously, the total investment over 40 years would be 100,000 but the amount available after 40 years (see earlier calculations) would be 319,452. Investing more, earlier is clearly better! This is what Ant would surely do.