Money Problems

Differential equations are the necessary tool for dealing with problems involving “continuously compounded” interest. Particularly in the cases of savings accounts, fixed rate bonds, annuities, mortgages, etc. the rules for change are precise so differential equations can be set up to predict future outcomes.

Many common investments grow at a rate proportional to their size (the statement “the rich get richer” has a lot of truth to it...the more you have, the more you have, the more your investment grows). Suppose you make an investment that increases in value by \( r \) percent per year “compounded continuously”. The compounded continuously means that over each interval of time \( t \) to \( t + \Delta t \), your investment will increase in value by some percent and after time \( t + \Delta t \) the amount you added will also grow by the same rule. Let \( M(t) \) be the money you have at time \( t \) (time in years). Then saying you get, say, 5 percent per year compounded continuously means that for each interval of time \( t \) to \( t + \Delta t \), your investment changes by

\[
M(t + \Delta t) = M(t) + 0.05 \Delta t M(t),
\]

here \( \Delta t \) is a fraction of a year so you get that fraction of 5 percent of your money added from \( t \) to \( t + \Delta t \). Doing a little algebra

\[
\frac{M(t + \Delta t) - M(t)}{\Delta t} = 0.05 M(t)
\]

and taking a limit as \( \Delta t \to 0 \) as in calculus 1, you get

\[
d\frac{M}{dt} = 0.05 M
\]

a differential equation for the size of your investment. If you start with \( M(0) = M_0 \), then the solution of the above equation is

\[
M(t) = M_0 e^{0.05 t}.
\]

In particular, after one year you have \( M(1) = e^{0.05} M_0 \). Recalling that

\[
e^{0.05 t} = 1 + 0.05 t + \frac{0.05^2}{2!} t^2 + ...
\]

(from the Taylor expansion of \( e^{0.05 t} \) at \( t = 0 \)) we see that

\[
M(1) \approx M_0 + 0.05 M_0 + \frac{0.05^2}{2!} M_0 + ...
\]

or a little more than 5 percent more than you started with—which is why continuous compounding gets you more than yearly (or some other time period) compounding.

EXERCISE: If you invest 5000 dollars at 5 percent per year compounded continuously, how long does it take for you to reach 1,000,000 dollars?

Given the answer above, you probably will want to add to your investment regularly...Suppose you have an investment that grows at 5 percent per year compounded continuously, you start
by investing 5000 dollars (so $M(0) = 5000$) and you add 5000 dollars per year to the account in many small additions. The new differential equations model is

$$\frac{dM}{dt} = 0.05M + 5000,$$

$M(0) = 5000$

This is an autonomous equation (so you can draw the phase line), a separable equation, and a linear equation (so you have a choice of techniques if you want to find a formula for the solution).

EXERCISE Draw the phase line, describe solutions qualitatively and find the general and particular solution for this initial value problem.

Once you have saved up a nest egg, you can set up an annuity, which pays you a fixed sum in many payments through the year. Suppose you have saved up 1,000,000 dollars and you wish to retire and remove 60,000 dollars per year (5000 per month) from this account. Suppose what remains in the account will still be invested at 5 percent per year compounded continuously. The new model is, $M(t)$ is the amount of money that remains at time $t$, $M(0) = 1,000,000$ and

$$\frac{dM}{dt} = 0.05M - 60,000$$

(technically this is an approximation since you only take money out monthly, not continuously, but it is pretty close).

EXERCISE How long will the money last?

Variations on this sort of problem are fixed rate mortgages (you borrow money and pay back a fixed amount each month), and other sorts of loans.

One last exercise...

EXERCISE Ant starts saving now with no money but promises to put 1000 per year in an investment that pays 5 percent per year compounded continuously.

Grasshopper doesn’t want to save now, but promises to start saving 20 years from now at of 2000 per year, also in an account that grows at 5 percent per year compounded continuously.

Both end up after 40 years investing the same amount of money, but who has more at the end of 40 years?

(Moral: If you want to retire eventually, start saving now, even if just a little bit!)