INSTRUCTIONS: Complete the following problem ON THE PAPER AND IN THE SPACE PROVIDED. Think about your answers carefully before writing. Neatness and clarity will count much more than usual.

You are allowed, and even encouraged to discuss these problems with classmates. (Particularly during class time on Thursday, but other times are OK.) However, you must abide by the following:

1. You must "write up" your own solutions. Talk with others in class to make sure you know how to do the problems and to help think about key points, but do not write your solutions while talking with another student. If you follow this rule, your papers will be your own—if you do not, then you will probably write solutions that are too similar.

2. Show your work. Solutions without work will receive no credit. (E.g., you should write down the characteristic polynomial and either the factorization or plug into the quadratic equation, don't just write down the eigenvalues out of the blue.)

3. Do NOT discuss the problems with anyone outside of class.

4. You MUST acknowledge who you work with. Keep a list of those you talk to and provide it in the space below.
1. Say everything you can about solutions of the forced harmonic oscillator equation

\[ \frac{d^2y}{dt^2} + a \frac{dy}{dt} + by = c \cos(\alpha t) \]

where \( a, b, c, \alpha \) are the right most non-zero digits of your ID number (e.g., if your ID number is U12-031-2056, then \( a = 1, b = 2, c = 5, \alpha = 6 \)).

What can you list -

For natural response -

Eigenvalues.

Classification (under, over, critically, damped)

Formula for natural response - natural period

Rate it goes to 0 (\( a > 0 \) so definitely damped)

Forced response: All solutions \( \rightarrow \) forced response

Forcing period

Formula for forced response

Amplitude of forced response

(is it close to resonance?)

Graph of typical \( y(t) \) solution.
2. Older watches operate via the oscillations of a spring, "ticking" each time the spring passes the "rest" state. As a well maintained watch gets older, will the watch run slow or fast? (State clearly what effect age will have on the spring in the watch and justify what effect this will have on how the watch keeps time.)

If the watch is well maintained then it's clear so changes come from wear in the metal - Eq.

"Tired" spring means spring constant k decreases.

\[
\text{For } \frac{d^2y}{dt^2} + \frac{k}{m} \frac{dy}{dt} + \frac{k}{m} y = 0
\]

\[\lambda = \frac{-b + \sqrt{b^2 - 4ac}}{2a}\]

\[\lambda = \frac{-\frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \frac{4}{2m^2}}}{2m} = \frac{-\frac{b}{2m} + \frac{\sqrt{4km - b^2}}{2m}}{2m} \]

Natural period:

\[T = \frac{2\pi}{4\pi \sqrt{4km - b^2}} = \frac{1}{2} \left(\frac{4\pi \sqrt{4km - b^2}}{2}\right) = \frac{1}{2} \left(\frac{4\pi \sqrt{4km - b^2}}{2}\right) = \frac{1}{2} \left(\frac{4\pi \sqrt{4km - b^2}}{2}\right) \]

So period increases, clock runs fast.

3. Sometimes when I am walking with a cup of coffee, it seems like the coffee just "jumps" out of the cup. How might a forced oscillator model explain this and what should I do to prevent this from happening?

Possibly the act of walking acts as periodic forcing on the sloshing coffee. If forcing period is not equal to natural period then amplitude of forced response gets large.

Test 1) Is it reasonable to model sloshing of liquid by harmonic oscillator?

2) Is walking period related to forcing period?
4. Consider the following forced harmonic oscillator

\[ \frac{d^2y}{dt^2} + 0.2 \frac{dy}{dt} + ay = 5 \cos(\sqrt{bt}) \]

where \( a = b \) is the right most non-zero digit of your ID number. Which is more effective in reducing the amplitude of the forced response, reducing the amplitude of the forcing to 3 or replacing the number \( a \) with 0.9a?

Amplitude of forced response:

\[ \frac{5}{(1.9a - a)^2 + 0.2^2 \left( \sqrt{a} \right)^2} \]

When \( b = a \), we get

\[ \frac{5}{(0.2^2 \sqrt{a})^2} = \frac{5}{0.04 \sqrt{a}} = 125 \sqrt{a} \]

For forcing amplitude 3, we get

forced response amplitude \( \frac{3}{0.04 \sqrt{a}} = \frac{75}{\sqrt{a}} \)

For \( \frac{d^2y}{dt^2} + 0.2 \frac{dy}{dt} + 0.9a y = 5 \cos(\sqrt{bt}) \)

we get forced response amplitude

\[ \frac{5}{(1.9a - a)^2 + 0.2^2 \left( \sqrt{a} \right)^2} \]

Thus plug in a and compare.
5. The equation modeling the torsional twisting of a suspension bridge is given by

\[ \frac{d^2\theta}{dt^2} = -\frac{6K}{m} \cos(\theta) \sin(\theta) - \delta \frac{d\theta}{dt} + f(t) \]

where the first term on the right is the "restoring" force of the cables, the second is the damping and the third is external forcing (say a gusty wind). For definiteness we take \( \delta = 0.01, \frac{6K}{m} = 2.4 \) and \( f(t) = 0.06 \sin(\omega t) \).

(a) The "standard" first step is to "linearize" this equation, i.e., note that \( \cos(\theta) = 1 - \frac{\theta^2}{2} \ldots \) and \( \sin(\theta) = \theta - \frac{\theta^3}{6} \ldots \), so use \( \cos(\theta) \approx 1 \) and \( \sin(\theta) \approx \theta \). Write the resulting equation and state under what assumptions this approximation would be justifiable.

\[ \frac{d^2\theta}{dt^2} = -\frac{6K}{m} \theta - \delta \frac{d\theta}{dt} + f(t) \]

To make this approximation we must neglect \( \theta^3 \) and higher order terms (\( \theta \approx 0 \)).

(b) For the linearized equation, for what value of \( \omega \) (in \( f(t) = 0.06 \sin(\omega t) \)) do you expect the largest amplitude forced response, and what is the amplitude of the forced response in this case?

Since \( \delta \) is very small, we are near resonance when \( \omega \approx \frac{\pi}{2} \) and in that case the amplitude of the forced response is

\[ \frac{0.06}{\sqrt{\left(0.01^2 + (\frac{\pi}{2})^2\right)^2 + 0.01^2 \times \frac{\pi}{2}^2}} = \frac{0.06}{0.01 \times \frac{\pi}{2}}. \]
(c) Now rewrite the full, unforced equation

\[
\frac{d^2 \theta}{dt^2} = -2.4 \cos(\theta) \sin(\theta) - 0.01 \frac{d\theta}{dt}
\]

as a linear system and find the equilibrium points?

Equilibria \( u = 0 \)

\[-2.4 \cos(\theta) \sin(\theta) - 0.01u = 0\]

So \( \cos(\theta) = 0 \) or \( \sin(\theta) = 0 \)

\( \theta = 0 \), \( \theta = \pm \pi, \pm 3\pi, \pm 5\pi, \ldots \)

(d) Sketch the direction field for this equation for \(-4 < \theta < 4\) (and appropriate range for the velocity) and sketch the solutions on the phase plane.

The phase plane shows solutions spiralling slowly to equilibria.