1. Find the general solution of

\[
\frac{dy}{dt} = -y + 3e^{-2t}
\]

This is a linear equation.

General solution: \( \frac{dy}{dt} = -y \) is \( y_h(t) = ke^{-t} \)

To find one solution of the forced equation, we guess \( y_p(t) = \alpha e^{-2t} \).

Plugging in, we get

\[-2\alpha e^{-2t} = -\alpha e^{-2t} + 3e^{-2t} \]

So \( -\alpha = 3 \) \( \Rightarrow \alpha = -3 \)

So \( y_p(t) = -3e^{-2t} \)

So the general solution is \( y(t) = ke^{-t} - 3e^{-2t} \)
2. Describe the long-term behavior of solutions of

\[ \frac{dy}{dt} = -4y + \cos(3t). \]

DO NOT SOLVE THE EQUATION (besides, that would take too long), just say what you can about solutions with minimal computations, your knowledge of linear equations and qualitative methods.

The natural response \( y_n(t) = k e^{-4t} \to 0 \text{ as } t \to \infty \)
So all solutions \( \to \) forced response as \( t \to \infty \).

The forced response has the form \( y_f(t) = \alpha \cos(3t) + \beta \sin(3t) \)
So it has period \( \frac{2\pi}{3} \) (and is periodic).

Since \( -1 \leq \cos(3t) \leq 1 \) the slope field is negative for \( y > \frac{1}{4} \) and positive for \( y < -\frac{1}{4} \).

So, the forced response has amplitude \( \leq \frac{1}{4} \).
3. Say what you can about the solution of

\[
\frac{dy}{dt} = -3y + b(t), \quad y(0) = 4
\]

where all we know about \(b(t)\) is that \(0 < b(t) < 6\) for all \(t\).

Again the natural response is

\[y_n(t) = ke^{-3t}\text{ which } \to 0 \text{ as } t \to \infty.\]

Because \(0 < b(t) < 6\) for all \(t\)

\[-3y + b(t) < 0 \quad \text{if } y > 2\]

and \[-3y + b(t) > 0 \quad \text{if } y < 0\]

So the solution with \(y(0) = 4\) will tend quickly to the forced response which is between \(0 < y < 2\).
4. Suppose you have a 10,000m³ pond has two streams, A and B, flowing in and one stream, C, flowing out. Suppose 1000m³ of water flows in per day from stream A and this water contains 1.0kg of road salt per cubic meter. The flow of stream B is 2000m³ per day and its water contains 0.25kg of road salt per cubic meter. The pond is well mixed and 3000m³ per day flows out stream C (so the volume of the pond is constant).

(a) Write down a differential equation model for the amount of road salt in the pond (be sure to define your variables).
   \[ \frac{dS}{dt} = 1000 \cdot 1 + 2000 \cdot 0.25 - 3000 \cdot \frac{S}{15000} \]
   or \[ \frac{dS}{dt} = 1500 - \frac{3S}{15000} \]

(b) Describe the long term behavior of the amount of road salt in the pond.
   This is a autonomous equation \[ 1500 - \frac{3S}{15000} = 0 \]
   \[ \frac{3S}{15000} = 1500 \]
   \[ S = 50000 \]
   \[ \frac{dS}{dt} < 0 \text{ for } S \text{ large} \]
   \[ \frac{dS}{dt} > 0 \text{ for } S \text{ small} \]
   One solution \[ S(0) = \text{5000} \]
   and \[ \lim_{t \to \infty} S(t) = \text{5000} \] regardless of \( S(0) \) value.

(c) If initially there is no road salt in the pond, ABOUT how long does it take the amount of salt to level off? (You only need to give a ROUGH estimate—days, weeks, months, years? But justify your answer.)
   Well \[ \frac{dS}{dt} = -\frac{3S}{1500} + 1500 \text{ is linear} \]

   So \[ S(t) = k e^{-3t} + S_o(t) \text{ and } S_0(t) = 5000 \]

   So for how long \( e^{-3t} \text{ is } e^{-3t} \approx 0 \)

   When \( t \approx 10, e^{-3t} \approx e^{-3} \)

   When \( t \approx 30, e^{-3t} \approx e^{-9} \text{ so it takes on the order of a month for } S(t) \text{ to level off.} \]
5. Consider the system

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= x - x^3 - \frac{y}{2}
\end{align*}
\]

(a) Find the equilibrium points for this system (Hint: there are 3).

\[
\begin{align*}
\text{Soln: } y &= 0 \\
\text{So } x - x^3 &= 0 \\
&\quad + x - X^2 - \frac{X}{2} = 0 \\
&\quad X (1 - X^2) = 0 \\
&\quad X = 0 \text{ or } X^2 = 1 \Rightarrow X = \pm 1
\end{align*}
\]

So (0, 0), (1, 0) and (-1, 0) are equilibrium points.

(b) The direction field for this system is shown below. Sketch the solution with initial point at \( t = 0 \) the large dot on the y axis. (You may do the sketch on the direction field picture, but be as accurate as you can.)

(c) Sketch the \( x(t) \) and \( y(t) \) graphs for the solution you drew in part (b) AND describe its behavior in a sentence.