A Taniyama product for the Riemann zeta function

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Abstract A Taniyama product for the Riemann zeta function is defined and an analogue of Mertens' theorem is proved.

1 Introduction

Tucked unobtrusively into Taniyama's memoir [4] on compatible families of ℓ-adic representations is a curious identity expressing the zeta function of such a family as an infinite product of imprimitive Artin L-functions ([4], p. 356, Theorem 3). The simplest case of the identity (arising from the cyclotomic character, or from its inverse, depending on one's conventions) is

$$\zeta(s-1)/\zeta(s) = \prod_{c \geqslant 1} \zeta_c(s), \tag{1}$$

where $\zeta(s)$ is the Riemann zeta function and $\zeta_c(s)$ is the imprimitive Dedekind zeta function – imprimitive because the Euler factors at the primes dividing c are removed – of the cyclotomic field K_c generated over $\mathbb{Q}$ by the cth roots of unity. Thus $\zeta_1(s)$ is $\zeta(s)$ itself, $\zeta_2(s)$ is $(1-2^{-s})\zeta(s)$, and so on. The infinite product converges for $\Re(s) > 2$.

In this note we modify (1) slightly so as to obtain a product for $\zeta(s)$ rather than $\zeta(s-1)/\zeta(s)$. Let K_c^+ be the maximal totally real subfield of K_c, and let $\zeta_c^+(s)$ be the Dedekind zeta function of K_c^+ with the Euler factors at the primes dividing c removed. Then

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$$\zeta(s) = \prod_{c \geqslant 1} \zeta_c^+(s+1). \tag{2}$$

Like the traditional Euler product

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}, \tag{3}$$

the Taniyama product (2) converges for $\Re(s) > 1$.

The main result of this note can be viewed as an analogue of Mertens' theorem [2]. It bears the same relation to (2) as Mertens' theorem does to (3), and Mertens' theorem itself figures prominently in the the proof. Let γ denote the Euler-Mascheroni constant.

Theorem 1. $\prod_{c \leqslant x} \zeta_c^+(2) \sim e^\gamma \log x$.

Of course $\zeta_c^+(2)$ can be computed explicitly in terms of generalized Bernoulli numbers. For a primitive Dirichlet character χ of conductor q, let

$$b_{2,\chi} = \sum_{j=1}^{q} \chi(j)(j^2/q - j + q/6).$$

Also write d_c^+ for the discriminant of K_c^+, and let $\varphi(c)$ be the cardinality of $(\mathbb{Z}/c\mathbb{Z})^\times$. For the sake of a succinct formula we put

$$\phi(c) = \begin{cases} \varphi(c) & \text{if } c \geqslant 3 \\ 2 & \text{if } c = 1 \text{ or } 2. \end{cases}$$

Then

$$\zeta_c^+(2) = \frac{\pi^{\phi(c)}}{d_c^{+\,3/2}} \prod_{q|c} \prod^{*}_{\substack{\chi \bmod q \\ \chi(-1)=1}} b_{2,\chi} \prod_{p|c} (1-\chi(p)p^{-2}), \tag{4}$$

where the asterisk indicates that in the second product, χ runs over *primitive* characters of conductor q.

Expressions similar to (4) have arisen in other contexts. For example, nearly the same triple product occurs in a formula of Yu [6] for the order of a certain cuspidal divisor class group of the modular curve $X_1(N)$ (see also Yang [5], p. 521). Even so, the differences between Yu's formula and (4) appear to be significant enough to preclude a straightforward interpretation of Theorem 1 as an asymptotic average of cuspidal divisor class numbers.

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2 Taniyama's identity

A proof of (1) is of course subsumed in Taniyama's proof of his general formula, but we will nonetheless sketch a proof here before going on to the modification (2). For a prime p not dividing c let $f(p,c)$ be the order of the residue class of p in $(\mathbb{Z}/c\mathbb{Z})^\times$. Also, write $Z(s)$ for the right-hand side of (1). Then $Z(s)$ can be written as the double product

$$Z(s) = \prod_{c\geqslant 1}\prod_{p\nmid c}(1-p^{-sf(p,c)})^{-\varphi(c)/f(p,c)}, \tag{5}$$

where the inner product is $\zeta_c(s)$ and runs over primes not dividing c. The proof of (1) amounts to reversing the order of multiplication in the double product in (5). By choosing a branch of $\log Z(s)$ we can do the computation additively, and the absolute convergence of the resulting triple sum in the right half-plane $\Re(s) > 2$ will show *a posteriori* that the original double product is meaningful in this region and that the calculation is legitimate.

We define our branch of $\log Z(s)$ by

$$\log Z(s) = \sum_{c\geqslant 1}\sum_{p\nmid c}\frac{\varphi(c)}{f(p,c)}\sum_{m\geqslant 1}\frac{p^{-mf(p,c)s}}{m}. \tag{6}$$

Putting $d = mf(p,c)$ and summing over $d \geqslant 1$, we obtain

$$\log Z(s) = \sum_{c\geqslant 1}\sum_{d\geqslant 1}\sum_{\substack{p\nmid c\\ f(p,c)|d}}\frac{\varphi(c)}{d}p^{-ds}$$

or equivalently (since $f(p,c)|d$ if and only if $p^d \equiv 1 \bmod c$)

$$\log Z(s) = \sum_{p}\sum_{d\geqslant 1}\sum_{\substack{c\geqslant 1\\ c|p^d-1}}\frac{\varphi(c)}{d}p^{-ds}.$$

As $\sum_{m|n}\varphi(m) = n$, we conclude that

$$\log Z(s) = \sum_{p}\sum_{d\geqslant 1}d^{-1}p^{-ds}(p^d-1).$$

The inner sum equals $\log((1-p^{1-s})^{-1}(1-p^{-s}))$, and (1) follows.

3 The modification

The proof of (2) is much the same. Put $\varphi^+(c) = [K_c^+ : \mathbb{Q}]$, so that $\varphi^+(c)$ is 1 if $c = 1$ or 2 and $\varphi(c)/2$ otherwise. (Note also that $\varphi^+(c) = \phi(c)/2$, where $\phi(c)$ is as in the introduction.) If p is a prime not dividing c, then the order in $\mathrm{Gal}(K_c^+/\mathbb{Q})$ of a Frobenius at p will be denoted $f^+(p,c)$. Of course $f^+(p,c)$ is also the order of the coset represented by p in the quotient of $(\mathbb{Z}/c\mathbb{Z})^\times$ by the image of $\{\pm 1\}$. Let $Z^+(s)$ denote the right-hand side of (2), and write $Z^+(s)$ as a double product:

$$Z^+(s) = \prod_{c \geqslant 1} \prod_{p \nmid c} (1 - p^{-sf^+(p,c)})^{-\varphi^+(c)/f^+(p,c)}. \tag{7}$$

Define a branch of $\log Z^+(s)$ by setting

$$\log Z^+(s) = \sum_{c \geqslant 1} \sum_{p \nmid c} \frac{\varphi^+(c)}{f^+(p,c)} \sum_{m \geqslant 1} \frac{p^{-mf^+(p,c)s}}{m}. \tag{8}$$

The calculation will again show that the triple sum is absolutely convergent for $\Re(s) > 2$. Putting $d = mf^+(p,c)$ and summing over $d \geqslant 1$, we find

$$\log Z^+(s) = \sum_{c \geqslant 1} \sum_{d \geqslant 1} \sum_{\substack{p \nmid c \\ f^+(p,c) | d}} \frac{\varphi^+(c)}{d} p^{-ds}$$

as before. But the condition $f^+(p,c)|d$ means $c|p^d - 1$ or $c|p^d + 1$, so we get

$$\log Z^+(s) = \sum_{p} \sum_{d \geqslant 1} \sum_{c | (p^d \pm 1)} \frac{\varphi^+(c)}{d} p^{-ds}. \tag{9}$$

We emphasize that the innermost sum is the sum over all c such that at least one of the conditions $c|p^d - 1$ and $c|p^d + 1$ is satisfied.

If $c \geqslant 3$ then the conditions $c|p^d - 1$ and $c|p^d + 1$ are mutually exclusive and $\varphi^+(c) = \varphi(c)/2$, so we have

$$\sum_{\substack{c \geqslant 3 \\ c|p^d \pm 1}} \varphi^+(c) = \frac{1}{2} \sum_{\substack{c \geqslant 3 \\ c|p^d - 1}} \varphi(c) + \frac{1}{2} \sum_{\substack{c \geqslant 3 \\ c|p^d + 1}} \varphi(c). \tag{10}$$

On the other hand, if $c = 1$ or 2 then the conditions $c|p^d - 1$ and $c|p^d + 1$ are both satisfied (for if $c = 2$ then p is odd), but $\varphi^+(c) = 1$. So equation (10) is correct without the restriction $c \geqslant 3$, and the identity $\sum_{m|n} \varphi(m) = n$ gives

$$\sum_{c|(p^d \pm 1)} \varphi^+(c) = \frac{1}{2}((p^d - 1) + (p^d + 1)) = p^d. \tag{11}$$

Multiplying through by p^{-ds}/d in (11) and inserting the result in (9), we obtain

$$\log Z^+(s) = \sum_p \log(1 - p^{1-s})^{-1},$$

or in other words

$$\prod_{c \geqslant 1} \zeta_c^+(s) = \zeta(s-1) \tag{12}$$

for $\Re(s) > 2$. Replacing s by $s+1$ gives (2) for $\Re(s) > 1$.

4 The analogue of Mertens' theorem

We shall prove that

$$\prod_{c \leqslant x+1} \zeta_c^+(2) \sim e^\gamma \log x. \tag{13}$$

Theorem 1 is an immediate consequence of (13), because $\log(x-1) \sim \log x$.

We proceed as in the derivation of (2), but with two crucial changes: first, we take $s = 2$, and second, c now runs over the *finite* set of positive integers $\leqslant x+1$. Thus (8) is replaced by

$$\log \prod_{c \leqslant x+1} \zeta_c^+(2) = \sum_{c \leqslant x+1} \sum_{p \nmid c} \frac{\varphi^+(c)}{f^+(p,c)} \sum_{m \geqslant 1} \frac{p^{-2mf^+(p,c)}}{m}. \tag{14}$$

Next we make the change of variables $d = mf^+(p,c)$. Since c runs over a finite set and the Dirichlet series for $\log \zeta_c^+(s)$ is absolutely convergent for $\Re(s) > 1$ and in particular for $s = 2$, we can rearrange the order of summation to obtain

$$\log \prod_{c \leqslant x+1} \zeta_c^+(2) = \sum_p \sum_{d \geqslant 1} \sum_{\substack{c \leqslant x+1 \\ c|p^d \pm 1}} \varphi^+(c) \frac{p^{-2d}}{d} \tag{15}$$

as in (9). For the sake of notational simplicity, we conflate the double sum over p and d into a single sum over p^d, and we put

$$\Phi(p^d,x) = \frac{p^{-2d}}{d} \sum_{\substack{c \leqslant x+1 \\ c \mid p^d \pm 1}} \varphi^+(c). \tag{16}$$

Then (15) can be written in the form

$$\log \prod_{c \leqslant x+1} \zeta_c^+(2) = \sum_{\substack{p^d > x \\ d \geqslant 2}} \Phi(p^d,x) + \sum_{p > x} \Phi(p,x) + \sum_{p^d \leqslant x} \Phi(p^d,x). \tag{17}$$

We shall prove the following assertions:

$$\sum_{\substack{p^d > x \\ d \geqslant 2}} \Phi(p^d,x) = o(1). \tag{18}$$

$$\sum_{p > x} \Phi(p,x) = o(1). \tag{19}$$

$$\sum_{p^d \leqslant x} \Phi(p^d,x) = \log \prod_{p \leqslant x} (1-p^{-1})^{-1} + o(1). \tag{20}$$

Granting these statements and using them in (17), we find that

$$\log \prod_{c \leqslant x+1} \zeta_c^+(2) = \log \prod_{p \leqslant x} (1-p^{-1})^{-1} + o(1),$$

whence exponentiation and an appeal to Mertens' theorem give (13).

To prove (18), we first note that

$$\sum_{\substack{c \leqslant x+1 \\ c \mid p^d \pm 1}} \varphi^+(c) \leqslant p^d. \tag{21}$$

by (11). Thus $\Phi(p^d,x) \leqslant p^{-d}/d$ by (16), whence the left-hand side of (18) is bounded by the sum of the terms with $p^d > x$ in the convergent double series $\sum_p \sum_{d \geqslant 2} p^{-d}/d$. Since the tail of a convergent series is $o(1)$, we obtain (18).

Next we prove (19). Take $x > 20$. It suffices to show that the sums

$$\sum\nolimits_1 = \sum_{x < p \leqslant x \log x} \Phi(p,x)$$

and

$$\sum\nolimits_2 = \sum_{p > x \log x} \Phi(p,x)$$

are both $o(1)$. Appealing once again to (21) and (16), we see that

$$\sum\nolimits_1 \leqslant \sum_{x<p\leqslant x\log x} p^{-1} = \log(\log(x\log x)/(\log x)) + o(1)$$

(cf. Chebyshev [1]). But $\log(\log(x\log x)/(\log x)) = \log(1+o(1))$, which is $o(1)$. Thus the sum $\sum_1$ is $o(1)$.

For $\sum_2$ we revert to an earlier order of summation:

$$\sum\nolimits_2 = \sum_{c\leqslant x+1} \varphi^+(c) \sum_{\substack{p\equiv\pm 1 \bmod c\\ p>x\log x}} p^{-2}. \tag{22}$$

We then rewrite the inner sum using Abel summation:

$$\sum_{\substack{p\equiv\pm 1 \bmod c\\ p>x\log x}} p^{-2} = \frac{\pi(y;c,\pm 1)}{y^2}\bigg|_{x\log x}^{\infty} + 2\int_{x\log x}^{\infty} \frac{\pi(y;c,\pm 1)}{y^3}\,dy, \tag{23}$$

where $\pi(y;c,\pm 1)$ is the number of primes $\leqslant y$ congruent to ± 1 mod c. By the strong form of the Brun-Titchmarsh theorem due to Montgomery and Vaughan [3], we have

$$\pi(y;c,\pm 1) \leqslant \frac{4y}{\varphi(c)\log(y/c)}. \tag{24}$$

Using (24) in (23) and then inserting the result in (22), we see that

$$\sum\nolimits_2 \leqslant \sum_{c\leqslant x+1} 8\int_{x\log x}^{\infty} \frac{dy}{y^2\log(y/c)},$$

the term $-\pi(x\log x;c,\pm 1)/(x\log x)^2$ having simply been omitted since it is negative. For $x > 20$ we have $(x\log x)/c > e$; hence the integrand is $\leqslant y^{-2}$ and the integal is $\leqslant (x\log x)^{-1}$. It follows that the sum over c is $\leqslant (1+1/x)/(\log x)$ and thus $o(1)$.

Finally we prove (20). The summation on the left-hand side of (20) is restricted to $p^d \leqslant x$, so if $c|p^d \pm 1$ then $c \leqslant x+1$. Hence $\Phi(p^d,x)$ coincides with p^{-d}/d by (11) and (16), so

$$\sum_{p^d\leqslant x} \Phi(p^d,x) = \sum_{p^d\leqslant x} p^{-d}/d.$$

We may write this identity as

$$\sum_{p^d\leqslant x} \Phi(p^d,x) = \sum_{p\leqslant x}\sum_{d\leqslant \frac{\log x}{\log p}} p^{-d}/d, \tag{25}$$

while

$$\log \prod_{p \leqslant x} (1 - p^{-1})^{-1} = \sum_{p \leqslant x} \sum_{d \geqslant 1} p^{-d}/d. \tag{26}$$

Subtracting (25) from (26), we see that

$$\log \prod_{p \leqslant x} (1 - p^{-1})^{-1} - \sum_{p^d \leqslant x} \Phi(p^d, x) = \sum_{p \leqslant x} \sum_{p^d > x} p^{-d}/d \tag{27}$$

If $p \leqslant x$ and $p^d > x$ then $d \geqslant 2$, so (27) gives

$$\log \prod_{p \leqslant x} (1 - p^{-1})^{-1} - \sum_{p^d \leqslant x} \Phi(p^d, x) \leqslant \sum_{\substack{p^d > x \\ d \geqslant 2}} p^{-d}/d.$$

The left-hand side is positive by (27), and as noted previously, the right-hand side is the tail of a convergent double series, and therefore $o(1)$. Hence the left-hand side is $o(1)$, and (20) follows.

5 The special value

For the sake of completeness, we recall the standard calculation of $\zeta_c^+(2)$ in terms of generalized Bernoulli numbers. Write $\zeta_c^+(s)$ as a product of Dirichlet L-functions associated to even Dirichlet characters to the modulus c:

$$\zeta_c^+(s) = \prod_{\substack{\chi \bmod c \\ \chi(-1)=1}} L(s, \chi). \tag{28}$$

We restrict attention to primitive characters by writing

$$\zeta_c^+(s) = \prod_{q|c} \prod_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} L(s, \chi) \prod_{p|c} (1 - \chi(p) p^{-s}). \tag{29}$$

Now recall the functional equation of $L(s, \chi)$: For χ even and primitive of conductor q, let

$$\Lambda(s, \chi) = q^{s/2} \pi^{-s/2} \Gamma(s/2) L(s, \chi); \tag{30}$$

then

$$\Lambda(s, \chi) = W(\chi) \Lambda(1 - s, \overline{\chi}), \tag{31}$$

where $W(\chi)$ is the root number of χ. On the other hand, according to a classic formula we have $L(1-k,\chi) = -b_{k,\chi}/k$ for integers $k \geqslant 2$ (and actually even for $k=1$ if $\chi \neq 1$). Taking $k=2$ and applying (31), we obtain

$$L(2,\chi) = \pi^2 b_{2,\overline{\chi}} W(\chi)/q^{3/2}. \tag{32}$$

Next recall that if χ has order $\geqslant 3$ then $W(\chi)W(\overline{\chi}) = 1$, while if $\chi^2 = 1$ then $W(\chi) = 1$. Thus on substituting (32) in (29), we obtain

$$\zeta_c^+(2) = \prod_{q|c} (\pi^2/q^{3/2})^{\psi^+(q)} \prod_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} b_{2,\chi} \prod_{p|c} (1-\chi(p)p^{-2}), \tag{33}$$

where $\psi^+(q)$ is the number of even Dirichlet characters which are primitive of conductor q. Since $\sum_{q|c} \psi^+(q) = \varphi^+(c)$ we have

$$\prod_{q|c} (\pi^2)^{\psi^+(q)} = \pi^{\phi(c)}. \tag{34}$$

Furthermore

$$\prod_{q|c} q^{\psi^+(q)} = d_c^+, \tag{35}$$

as one sees, for example, by observing that the exponential factor in the functional equation of the Dedekind zeta function of K_c^+ is $(d_c^+)^{s/2}$, while the exponential factor in (31) or rather (30) is $q^{s/2}$. On substituting (34) and (35) in (33), we obtain (4).

6 A question

For $c \geqslant 3$, let ξ_c be the quadratic Hecke character of K_c^+ associated to the extension K_c/K_c^+, and let $L(s,\xi_c)$ be the corresponding Hecke L-function. Write $L_c(s)$ for the imprimitive Hecke L-function obtained from $L(s,\xi_c)$ by deleting the Euler factors at the primes dividing c. Also put $L_c(s) = 1$ for $c = 1$ or 2. Then $L_c(s) = \zeta_c(s)/\zeta_c^+(s)$ in all cases. Hence combining (1) with (12), we obtain

$$\zeta(s)^{-1} = \prod_{c \geqslant 1} L_c(s).$$

The infinite product converges for $\Re(s) > 2$, but $1/\zeta(s)$ is holomorphic and nonvanishing for $\Re(s) > 1$. Is the true region of convergence perhaps much larger than $\Re(s) > 2$? We can offer only a minimal enlargement:

Theorem 2. *The product* $\prod_{c\geqslant 1} L_c(s)$ *converges to* $\zeta(s)^{-1}$ *for* $\Re(s)\geqslant 2$.

Proof. For integers $c\geqslant 3$ and primes $p\nmid c$, put $\kappa(p,c)=\xi_c(\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of K_c^+ lying above p. If $c=1$ or 2 put $\kappa(p,c)=0$. Then

$$L_c(s)=\prod_{p\nmid c}(1-\kappa(p,c)p^{-f^+(p,c)s})^{-\varphi^+(p,c)/f^+(p,c)}$$

for $\Re(s)>1$, and consequently

$$\log\prod_{c\leqslant x+1} L_c(s)=\sum_{c\leqslant x+1}\sum_{p\nmid c}\frac{\varphi^+(c)}{f^+(p,c)}\sum_{m\geqslant 1}\kappa(p,c)^m\,\frac{p^{-mf^+(p,c)s}}{m}.$$

Making the change of variables $d=mf^+(p,c)$ as before, we obtain

$$\log\prod_{c\leqslant x+1} L_c^+(s)=\sum_{p}\sum_{d\geqslant 1}\sum_{\substack{c\leqslant x+1\\ c|p^d\pm 1}}\varphi^+(c)\,\kappa(p,c)^{d/f^+(p,c)}\,\frac{p^{-ds}}{d}$$

(note that the condition $c|p^d\pm 1$ means precisely that $f^+(p,c)|d$). All of this is valid for $\Re(s)>1$, but we now assume that $\Re(s)\geqslant 2$ or simply that $\Re(s)=2$, since the case $\Re(s)>2$ has already been dealt with. Put

$$\Psi(p^d,x,s)=\frac{p^{-ds}}{d}\sum_{\substack{c\leqslant x+1\\ c|p^d\pm 1}}\varphi^+(c)\,\kappa(p,c)^{d/f^+(p,c)}. \tag{36}$$

Comparing (36) with (16), we see that

$$|\Psi(p^d,x,s)|\leqslant\Phi(p^d,x). \tag{37}$$

This relation will largely reduce the proof to our previous estimates. Indeed, as in (17), we can write

$$\log\prod_{c\leqslant x+1} L_c(s)=\sum_{\substack{p^d>x\\ d\geqslant 2}}\Psi(p^d,x,s)+\sum_{p>x}\Psi(p,x,s)+\sum_{p^d\leqslant x}\Psi(p^d,x,s),$$

and the first and second sums on the right-hand side are $o(1)$ by (18), (19), and (37). Thus to prove the theorem it suffices to show that

$$\sum_{p^d\leqslant x}\Psi(p^d,x,s)=\log\zeta(s)^{-1}+o(1). \tag{38}$$

The argument will be similar to the argument for (20).

The first point is that the condition $p^d\leqslant x$ in (38) renders the condition $c\leqslant x+1$ superfluous in (36). Thus in the context of (38) we have

$$\Psi(p^d,x,s)=\frac{p^{-ds}}{d}\sum_{c|p^d\pm 1}\varphi^+(c)\,\kappa(p,c)^{d/f^+(p,c)},\tag{39}$$

Now suppose that $c|p^d\pm 1$ (in other words that $f^+(p,c)|d$) and that $c\geqslant 3$. The conditions $p^d\equiv -1 \bmod c$ and $\kappa(p,c)^{d/f^+(p,c)}=-1$ are equivalent, because both are equivalent to the assertion that $f(p,c)=2f^+(p,c)$ and $d/f^+(p,c)$ is odd. It follows that the complementary conditions, namely $p^d\equiv 1 \bmod c$ and $\kappa(p,c)^{d/f^+(p,c)}=1$, are also equivalent, whence

$$\sum_{\substack{c\geqslant 3\\ c|p^d\pm 1}}\varphi^+(c)\,\kappa(p,c)^{d/f^+(p,c)}=\frac{1}{2}\sum_{\substack{c\geqslant 3\\ c|p^d-1}}\varphi(c)-\frac{1}{2}\sum_{\substack{c\geqslant 3\\ c|p^d+1}}\varphi(c).\tag{40}$$

As before, the restriction $c\geqslant 3$ can be eliminated throughout (40), because if $c=1$ or 2 then $\kappa(p,c)=0$.

With the restriction $c\geqslant 3$ removed, (40) implies that

$$\sum_{c|p^d\pm 1}\varphi^+(c)\,\kappa(p,c)^{d/f^+(p,c)}=\frac{1}{2}((p^d-1)-(p^d+1))=-1,$$

and therefore (39) gives

$$\sum_{p^d\leqslant x}\Psi(p^d,x,s)=-\sum_{p\leqslant x}\sum_{d\leqslant\frac{\log x}{\log p}}\frac{p^{-ds}}{d}.\tag{41}$$

On the other hand,

$$\log\zeta(s)^{-1}=-\sum_{p}\sum_{d\geqslant 1}\frac{p^{-ds}}{d}\tag{42}$$

Taking the absolute value of the difference of (41) and (42), we find

$$|\sum_{p^d\leqslant x}\Psi(p^d,x,s)-\zeta(s)^{-1}|\leqslant\sum_{p^d>x}\frac{p^{-ds}}{d}.$$

Since $\Re(s)\geqslant 2$ the right-hand side is the tail of a convergent series (namely the Dirichlet series for $\log\zeta(s)$) and is therefore $o(1)$. Thus (38) follows and the proof of the theorem is complete.

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