Hilbert modular forms and the Gross–Stark conjecture

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Abstract

Let F be a totally real field and χ an abelian totally odd character of F. In 1988, Gross stated a p-adic analogue of Stark's conjecture that relates the value of the derivative of the p-adic L-function associated to χ and the p-adic logarithm of a p-unit in the extension of F cut out by χ. In this paper we prove Gross's conjecture when F is a real quadratic field and χ is a narrow ring class character. The main result also applies to general totally real fields for which Leopoldt's conjecture holds, assuming that either there are at least two primes above p in F, or that a certain condition relating the $\mathscr{L}$-invariants of χ and χ^{-1} holds. This condition on $\mathscr{L}$-invariants is always satisfied when χ is quadratic.

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Introduction

Let F be a totally real field of degree n, and let

$$\chi : G_F := \mathrm{Gal}(\bar{F}/F) \to \overline{\mathbf{Q}}^{\times}$$

be a character of conductor $\mathfrak{n}$. Such a character cuts out a finite cyclic extension H of F, and can be viewed as a function on the ideals of F in the usual way, by setting $\chi(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not prime to $\mathfrak{n}$. Let $\mathrm{N}\mathfrak{a} = \mathrm{Norm}_{F/\mathbf{Q}}(\mathfrak{a})$ denote the norm of $\mathfrak{a}$. Fix a rational prime p and a choice of embeddings $\overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}_p \subset \mathbf{C}$ that will remain in effect throughout this article. The character χ may be viewed as having values in $\overline{\mathbf{Q}}_p$ or $\mathbf{C}$ via these embeddings.

Let S be any finite set of places of F containing all the archimedean places. Associated to χ is the complex L-function

$$L_S(\chi, s) := \sum_{(\mathfrak{a},S)=1} \chi(\mathfrak{a})\, \mathrm{N}\mathfrak{a}^{-s} = \prod_{\mathfrak{p}\notin S} (1 - \chi(\mathfrak{p})\, \mathrm{N}\mathfrak{p}^{-s})^{-1}, \tag{1}$$

which converges for $\mathrm{Re}(s) > 1$ and has a holomorphic continuation to all of $\mathbf{C}$ when $\chi \neq 1$. By work of Siegel [13], the value $L_S(\chi, n)$ is algebraic for each $n \leq 0$. (See for instance the discussion in §2 of [8], where $L_S(\chi, n)$ is denoted $a_S(\chi, n)$.)

Let E be a finite extension of $\mathbf{Q}_p$ containing the values of the character χ. Let

$$\omega : \mathrm{Gal}(F(\mu_{2p})/F) \to (\mathbf{Z}/2p)^{\times} \to \mathbf{Z}_p^{\times}$$

denote the p-adic Teichmuller character. If S contains all the primes above p, Deligne and Ribet [3] have proved the existence of a continuous E-valued function $L_{S,p}(\chi\omega, s)$ of a variable $s \in \mathbf{Z}_p$ characterized by the interpolation property

$$L_{S,p}(\chi\omega, n) = L_S(\chi\omega^n, n) \qquad \text{for all integers } n \leq 0. \tag{2}$$

The function $L_{S,p}(\chi\omega, s)$ is meromorphic on $\mathbf{Z}_p$, regular outside $s = 1$, and regular everywhere when $\chi\omega$ is non-trivial.

If $\mathfrak{p} \in S$ is any non-archimedean prime, and $R := S - \{\mathfrak{p}\}$, then

$$L_S(\chi, 0) = (1 - \chi(\mathfrak{p}))L_R(\chi, 0).$$

In particular, $L_S(\chi, s)$ vanishes at $s = 0$ when $\chi(\mathfrak{p}) = 1$, and equation (2) implies that the same is true of the p-adic L-function $L_{S,p}(\chi\omega, 0)$. Assume for the remainder of this article that the hypothesis $\chi(\mathfrak{p}) = 1$ is satisfied.

For $x \in \mathbf{Z}_p^{\times}$, let $\langle x \rangle = x/\omega(x) \in 1 + p\mathbf{Z}_p$. If $\mathfrak{p}$ does not divide p, the formula

$$L_{S,p}(\chi\omega, s) = (1 - \langle \mathrm{N}\mathfrak{p} \rangle^{-s})L_{R,p}(\chi\omega, s)$$

implies that

$$L'_{S,p}(\chi\omega, 0) = \log_p(\mathrm{N}\mathfrak{p}) L_R(\chi, 0), \tag{3}$$

where $\log_p : \mathbf{Q}_p^\times \to \mathbf{Q}_p$ denotes the usual Iwasawa p-adic logarithm.

If $\mathfrak{p}$ divides p, which we assume for the remainder of this article, a formula analogous to (3) comparing $L'_{S,p}(\chi\omega, 0)$ and $L_R(\chi, 0)$ has been conjectured in [8]. This formula involves the group $\mathcal{O}_{H,S}^\times$ of S-integers of H—more precisely, the χ^{-1}-component U_χ of the E-vector space $\mathcal{O}_{H,S}^\times \otimes E$:

$$U_\chi := (\mathcal{O}_{H,S}^\times \otimes E)^{\chi^{-1}} := \left\{ u \in \mathcal{O}_{H,S}^\times \otimes E \text{ such that } \sigma u = \chi^{-1}(\sigma) u \right\}. \tag{4}$$

Dirichlet's unit theorem implies that U_χ is a finite-dimensional E-vector space and that

$$\begin{aligned} \dim_E U_\chi &= \#\{v \in S \text{ such that } \chi(v) = 1\} \\ &= \mathrm{ord}_{s=0} L_S(\chi, s) \\ &= \mathrm{ord}_{s=0} L_R(\chi, s) + 1. \end{aligned}$$

In particular, the space U_χ is one-dimensional if and only if $L_R(\chi, 0) \neq 0$. Assume that this is the case, and let u_χ be any non-zero vector in U_χ.

The choice of a prime $\mathfrak{P}$ of H lying above $\mathfrak{p}$ determines two $\mathbf{Z}$-module homomorphisms

$$\mathrm{ord}_{\mathfrak{P}} : \mathcal{O}_{H,S}^\times \to \mathbf{Z}, \qquad \mathbf{L}_{\mathfrak{P}} : \mathcal{O}_{H,S}^\times \to \mathbf{Z}_p, \tag{5}$$

where the latter is defined by

$$\mathbf{L}_{\mathfrak{P}}(u) := \log_p(\mathrm{Norm}_{H_{\mathfrak{P}}/\mathbf{Q}_p}(u)). \tag{6}$$

Let $\mathrm{ord}_{\mathfrak{P}}$ and $\mathbf{L}_{\mathfrak{P}}$ also denote the homomorphisms from U_χ to E obtained by extending scalars to E. Following Greenberg (cf. equation (4') of [6] in the case $F = \mathbf{Q}$), the $\mathscr{L}$-invariant attached to χ is defined to be the ratio

$$\mathscr{L}(\chi) := -\frac{\mathbf{L}_{\mathfrak{P}}(u_\chi)}{\mathrm{ord}_{\mathfrak{P}}(u_\chi)} \in E. \tag{7}$$

This $\mathscr{L}$-invariant is independent of the choice of non-zero vector $u_\chi \in U_\chi$, and it is also independent of the choice of the prime $\mathfrak{P}$ lying above $\mathfrak{p}$. When $L_R(\chi, 0) = 0$, we (arbitrarily) assign the value of 1 to $\mathscr{L}(\chi)$.

The following is conjectured in [8] (cf. Proposition 3.8 and Conjecture 3.13 of loc. cit.):

Conjecture 1 (Gross). *For all characters χ of F and all $S = R \cup \{\mathfrak{p}\}$, we have*

$$L'_{S,p}(\chi\omega, 0) = \mathscr{L}(\chi) L_R(\chi, 0). \tag{8}$$

When $L_R(\chi, 0) = 0$, Conjecture 1 amounts to the statement $L'_{S,p}(\chi\omega, 0) = 0$. As explained in Section 1, this case of the conjecture follows from Wiles' proof of the Main Conjecture for totally real fields (assuming that χ is of "type S"; see Lemma 1.2). We will therefore assume that $L_R(\chi,0)\neq 0$. In this setting, Gross's conjecture suggests defining the *analytic $\mathscr{L}$-invariant of* χ by the formula

$$\mathscr{L}_{\rm an}(\chi) := \frac{L'_{S,p}(\chi\omega,0)}{L_R(\chi,0)} = \frac{d}{dk}\mathscr{L}_{\rm an}(\chi,k)_{k=1}, \tag{9}$$

where

$$\mathscr{L}_{\rm an}(\chi,k) := \frac{-L_{S,p}(\chi\omega,1-k)}{L_R(\chi,0)}. \tag{10}$$

Conjecture 1 can then be rephrased as the equality $\mathscr{L}(\chi)=\mathscr{L}_{\rm an}(\chi)$ between algebraic and analytic $\mathscr{L}$-invariants. The main result of this paper is:

Theorem 2. *Assume that Leopoldt's conjecture holds for F.*

1. *If there are at least two primes of F lying above p, then Conjecture 1 holds for all χ.*
2. *If $\mathfrak{p}$ is the only prime of F lying above p, assume further that*

$$\mathrm{ord}_{k=1}(\mathscr{L}_{\rm an}(\chi,k)+\mathscr{L}_{\rm an}(\chi^{-1},k)) = \mathrm{ord}_{k=1}\mathscr{L}_{\rm an}(\chi^{-1},k). \tag{11}$$

Then Conjecture 1 holds for both χ and χ^{-1}.

Remark 3. The somewhat mysterious condition formulated in (11) makes no *a priori* assumption on the order of vanishing of $L_{S,p}(\chi\omega,s)$ at $s=0$. It is automatically satisfied (after possibly interchanging χ and χ^{-1}) when $\mathscr{L}_{\rm an}(\chi,k)$ and $\mathscr{L}_{\rm an}(\chi^{-1},k)$ have different orders of vanishing. When these orders of vanishing agree, it stipulates that the sum of the leading terms at $k=1$ of $\mathscr{L}_{\rm an}(\chi,k)$ and $\mathscr{L}_{\rm an}(\chi^{-1},k)$ should be nonzero. In the setting that we are considering, where $L_R(\chi,0)\neq 0$, it is expected that the functions $\mathscr{L}_{\rm an}(\chi,k)$ and $\mathscr{L}_{\rm an}(\chi^{-1},k)$ both vanish to order 1 at $k=1$. If this is true, condition (11) amounts to the condition

$$\mathscr{L}_{\rm an}(\chi)+\mathscr{L}_{\rm an}(\chi^{-1})\neq 0. \tag{12}$$

One can show using the methods of section 4 that when $\mathfrak{p}$ is the unique prime of F lying above p and Leopoldt's conjecture holds for F, we have

$$\mathscr{L}(\chi)\neq 0 \text{ and } \mathscr{L}(\chi^{-1})\neq 0 \Longrightarrow \mathscr{L}(\chi)+\mathscr{L}(\chi^{-1})\neq 0.$$

Therefore, condition (12) is always expected to hold. The condition on the non-vanishing of the algebraic $\mathscr{L}$-invariant attached to χ may however be quite deep, and condition (11) appears to be a substantial hypothesis in the formulation of Theorem 2.

Remark 3 notwithstanding, Theorem 2 leads to the following two unconditional results.

Corollary 4. *Let F be a real quadratic field, and let χ be a narrow ring class character of F. Then Conjecture 1 holds for χ.*

Proof. Corollary 4 is unconditional because Leopoldt's conjecture is trivial for real quadratic F. Furthermore, since χ is a ring class character, the representations induced from F to $\mathbf{Q}$ by χ and χ^{-1} are equal and therefore the L-functions (both classical and p-adic) attached to these characters agree. It follows that the leading terms of $\mathscr{L}_{\mathrm{an}}(\chi, k)$ and $\mathscr{L}_{\mathrm{an}}(\chi^{-1}, k)$ are equal, and therefore condition (11) is satisfied. Corollary 4 follows. ☐

Corollary 5. *Let F be a totally real field satisfying Leopoldt's conjecture, and let χ be a narrow ray class character of F. Then Conjecture 1 holds for χ in either of the following two cases:*

1. *There are at least two primes of F above the rational prime p, or*
2. *The character χ is quadratic.*

Proof. The first case is direct consequence of Theorem 2, and the second follows from the fact that $\mathscr{L}_{\mathrm{an}}(\chi, k) = \mathscr{L}_{\mathrm{an}}(\chi^{-1}, k)$ when χ is quadratic. ☐

Remark 6. When $F_{\mathfrak{p}} = \mathbf{Q}_p$, Conjecture 1 leads to a p-adic analytic construction of non-trivial $\mathfrak{p}$-units in abelian extensions of F by exponentiating the first derivatives of the appropriate partial p-adic L-series. In this way, Conjecture 1 supplies a p-adic solution to Hilbert's twelfth problem for certain abelian extensions of F, just like Stark's original archimedean conjectures. See [8, Proposition 3.14] for a more detailed discussion of the application of Conjecture 1 to the analytic construction of class fields.

Remark 7. Conjecture 1 has been proved in [8, §4] in the case $F = \mathbf{Q}$ using an explicit expression for the Gross-Stark unit u_χ in terms of Gauss sums, which are related to values of the p-adic Gamma function by the Gross-Koblitz formula. The p-adic Gamma function is in turn related to the p-adic L-functions over $\mathbf{Q}$ by the work of Ferrero and Greenberg [4]. In contrast, the approach we have followed to handle more general totally real F does not construct the Gross-Stark unit u_χ directly. Instead, it exploits the two-dimensional p-adic representations attached to certain families of Hilbert modular forms to construct an annihilator for u_χ under the local Tate pairing. The explicit construction of these families allows us to relate these annihilators, and hence u_χ itself, to p-adic L-functions and to $\mathscr{L}_{\mathrm{an}}(\chi)$.

We now present a more detailed outline of the strategy used to prove Theorem 2, and give an overview of the contents of this article.

Cohomological interpretation of Conjecture 1

Let

$$\epsilon_{\mathrm{cyc}} : G_F \to \mathbf{Z}_p^\times \tag{13}$$

denote the cyclotomic character defined by

$$\sigma(\zeta) =: \zeta^{\epsilon_{\mathrm{cyc}}(\sigma)} \tag{14}$$

for any p-power root of unity $\zeta \in \overline{F}$. Write $E(\chi)(1)$ for the one-dimensional E-vector space equipped with the continuous action of G_F via $\chi\epsilon_{\mathrm{cyc}}$, and let $E(\chi^{-1})$ denote its Kummer dual, on which G_F acts via χ^{-1}. The first step is to exploit Tate's local duality and the reciprocity law of global class field theory to give an alternate description of $\mathscr{L}(\chi)$, following Greenberg [6]. This alternate description involves the subgroup $H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$ of the global cohomology group $H^1(F, E(\chi^{-1}))$ consisting of (continuous) classes whose restrictions to the inertia subgroups $I_{\mathfrak{q}} \subset G_F$ are unramified for all primes $\mathfrak{q} \neq \mathfrak{p}$ of F. Under the assumptions $L_R(\chi, 0) \neq 0$ and $\chi(\mathfrak{p}) = 1$, we show that

$$\dim_E H^1_{\mathfrak{p}}(F, E(\chi^{-1})) = [F_{\mathfrak{p}} : \mathbf{Q}_p], \qquad \dim_E H^1(F_{\mathfrak{p}}, E(\chi^{-1})) = [F_{\mathfrak{p}} : \mathbf{Q}_p] + 1,$$

and that the natural restriction map

$$H^1_{\mathfrak{p}}(F, E(\chi^{-1})) \longrightarrow H^1(F_{\mathfrak{p}}, E(\chi^{-1})) \tag{15}$$

is injective.

Since $\chi(\mathfrak{p}) = 1$, the group $H^1(F_{\mathfrak{p}}, E(\chi^{-1})) = H^1(F_{\mathfrak{p}}, E) = \mathrm{Hom}_{\mathrm{cts}}(G_{F_{\mathfrak{p}}}, E)$ contains two distinguished elements: the unique unramified homomorphism

$$\kappa_{\mathrm{nr}} \in \mathrm{Hom}(\mathrm{Gal}(F_{\mathfrak{p}}^{\mathrm{nr}}/F_{\mathfrak{p}}), \mathcal{O}_E)$$

sending the Frobenius element $\mathrm{Frob}_{\mathfrak{p}}$ to 1, and the restriction to $G_{F_{\mathfrak{p}}}$ of the p-adic logarithm of the cyclotomic character:

$$\kappa_{\mathrm{cyc}} := \log_p(\epsilon_{\mathrm{cyc}}) \in \mathrm{Hom}(G_F, E) = H^1(F, E). \tag{16}$$

Let $H^1(F_{\mathfrak{p}}, E)^{\mathrm{cyc}}$ denote the two-dimensional subspace of $H^1(F_{\mathfrak{p}}, E)$ spanned by κ_{nr} and κ_{cyc}, and let $H^1_{\mathfrak{p}}(F, E(\chi^{-1}))^{\mathrm{cyc}}$ denote its inverse image in $H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$ under the restriction map at $\mathfrak{p}$. It is proved in Section 1 that

$$\dim_E H^1_{\mathfrak{p}}(F, E(\chi^{-1}))^{\mathrm{cyc}} = 1.$$

If $\kappa \in H^1_{\mathfrak{p}}(F, E(\chi^{-1}))^{\mathrm{cyc}} \subset H^1(F_{\mathfrak{p}}, E)^{\mathrm{cyc}}$ is any non-zero class, we may thus write

$$\mathrm{res}_{\mathfrak{p}}(\kappa) =: x \cdot \kappa_{\mathrm{nr}} + y \cdot \kappa_{\mathrm{cyc}}, \qquad \text{with } x, y \in E.$$

The ratio x/y—the "slope" of the global line relative to the natural basis $(\kappa_{\mathrm{nr}}, \kappa_{\mathrm{cyc}})$—does not depend on the choice of κ. The main result of Section 1 is that $y \neq 0$ and that

$$\mathscr{L}(\chi) = -x/y.$$

Thanks to this result, the problem of proving Theorem 2 is transformed into the problem of constructing a global cohomology class $\kappa \in H^1_{\mathfrak{p}}(F, E(\chi^{-1}))^{\mathrm{cyc}}$ whose "coordinates" x and y can be computed explicitly and related to p-adic L-functions at 0—more precisely, such that

$$\mathrm{res}_{\mathfrak{p}}(\kappa) = -\mathscr{L}_{\mathrm{an}}(\chi) \cdot \kappa_{\mathrm{nr}} + \kappa_{\mathrm{cyc}}. \tag{17}$$

Construction of a cusp form

The construction of κ borrows heavily from the techniques initiated in [11] and extended and developed in [17] to prove the main conjecture of Iwasawa theory for totally real fields. We briefly outline the main steps in the notationally simpler case where $F = \mathbf{Q}$ and the prime p is odd. Let $m \geq 1$ denote the conductor of the odd Dirichlet character χ. Let R be the set of primes dividing $m\infty$, and let $S = R \cup \{p\}$.

For an integer $k \geq 1$, denote by $M_k(m, \chi) = M_k(\Gamma_1(m), \chi)$ the space of classical modular forms of weight k, level m and character χ with Fourier coefficients in E. Let (η, ψ) be a pair of (not necessarily primitive) Dirichlet characters of modulus m_η and m_ψ, respectively, such that $\eta\psi(-1) = (-1)^k$. A key role is played in the argument by the weight k Eisenstein series $E_k(\eta, \psi)$. After recalling the definitions of Hilbert modular forms and their q-expansions in Section 2.1, the Eisenstein series for the Hilbert modular group are defined in Section 2.2. A key result in that section is the calculation of their constant terms at certain cusps. When $F = \mathbf{Q}$, the Eisenstein series $E_k(\eta, \psi) \in M_k(m_\eta m_\psi, \eta\psi)$ for $k \geq 1$ and $(k, \eta, \psi) \neq (2, 1, 1)$ are given by:

$$E_k(\eta, \psi) := C_k(\eta, \psi) + \sum_{n=1}^{\infty} \Big(\sum_{d|n} \eta\left(\frac{n}{d}\right) \psi(d) d^{k-1} \Big) q^n, \tag{18}$$

where

$$C_k(\eta, \psi) := \frac{1}{2} \begin{cases} L(\psi, 1-k) & \text{if } \eta = 1; \\ L(\eta, 1-k) & \text{if } \psi = 1; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the Eisenstein series

$$\begin{aligned} E_1(1, \chi) &:= \frac{1}{2} L_R(\chi, 0) + \sum_{n=1}^{\infty} \Big(\sum_{d|n} \chi(d) \Big) q^n \\ E_{k-1}(1, \omega^{1-k}) &:= \frac{1}{2} L(\omega^{1-k}, 2-k) + \sum_{n=1}^{\infty} \Big(\sum_{d|n} \omega^{1-k}(d) d^{k-2} \Big) q^n \\ &= \frac{1}{2} \zeta_p(2-k) + \sum_{n=1}^{\infty} \Big(\sum_{d|n} \omega^{-1}(d) \langle d \rangle^{k-2} \Big) q^n \end{aligned} \tag{19}$$

belong to the spaces $M_1(m, \chi)$ and $M_{k-1}(p, \omega^{1-k})$ respectively. Here, $\zeta_p(s) = L_p(1, s)$ denotes the p-adic zeta-function of Kubota-Leopoldt. In (19), the character ω^{1-k} is always viewed as having modulus p, even when $k \equiv 1 \pmod{p-1}$. Let

$$G_{k-1}(1, \omega^{1-k}) := 2\zeta_p(2-k)^{-1} E_{k-1}(1, \omega^{1-k}) \tag{20}$$

$$= 1 + 2\zeta_p(2-k)^{-1} \sum_{n=1}^{\infty} \Big(\sum_{d|n} \omega^{-1}(d) \langle d \rangle^{k-2} \Big) q^n \tag{21}$$

be the associated normalized Eisenstein series of weight $k-1$.

In Section 2.3, we consider the product

$$P_k := E_1(1,\chi)G_{k-1}(1,\omega^{1-k}) \in M_k(mp,\chi\omega^{1-k}). \tag{22}$$

As explained in Section 2.3 in the setting of Hilbert modular forms, it follows from general principles that the series P_k can be expressed uniquely as the sum of a cusp form and a linear combination of the Eisenstein series (18) in $M_k(mp,\chi\omega^{1-k})$:

$$P_k = \left(\begin{array}{c}\text{Cusp}\\ \text{form}\end{array}\right) + \sum_{(\eta,\psi)\in J} a_k(\eta,\psi)E_k(\eta,\psi), \tag{23}$$

where (η,ψ) ranges over a set J of pairs of (not necessarily primitive) Dirichlet characters of modulus m_η and m_ψ, respectively, satisfying

$$m_\eta m_\psi = mp, \qquad \eta\psi = \chi\omega^{1-k}. \tag{24}$$

The main result of Section 2.3 is the computation of certain coefficients in (23) for $k > 2$:

$$a_k(1,\chi\omega^{1-k}) = -\mathscr{L}_{\mathrm{an}}(\chi,k)^{-1}, \qquad a_k(\chi,\omega^{1-k}) = -\mathscr{L}_{\mathrm{an}}(\chi^{-1},k)^{-1}\langle m\rangle^{k-1}, \tag{25}$$

where $\mathscr{L}_{\mathrm{an}}(\chi,k)$ is the quantity defined in (10). The derivation of (25) proceeds by comparing the constant terms of both sides of (23) at various cusps.

In Section 2.4, we consider Hida's idempotent

$$e : M_k(mp,\chi\omega^{1-k}) \longrightarrow M_k(mp,\chi\omega^{1-k}),$$

which is defined as

$$e := \lim U_p^{n!}$$

on the submodule of modular forms with Fourier coefficients in $\mathcal{O}_E$, and extended to the space $M_k(mp,\chi\omega^{1-k})$ by E-linearity. The image of e,

$$M_k^o(mp,\chi\omega^{1-k}) := eM_k(mp,\chi\omega^{1-k}),$$

has dimension that is bounded independently of k, and is called the *ordinary subspace*. The operator e preserves the space of cusp forms. For all (η,ψ) satisfying (24), and $k > 1$, it can be checked that

$$eE_k(\eta,\psi) = \begin{cases} E_k(\eta,\psi) & \text{if } p \nmid m_\eta; \\ 0 & \text{if } p \mid m_\eta. \end{cases}$$

Hence the modular form $P_k^o := eP_k$ can be written

$$P_k^o = \left(\begin{array}{c}\text{A cusp}\\ \text{form}\end{array}\right) + \sum_{(\eta,\psi)\in J^o} a_k(\eta,\psi)E_k(\eta,\psi), \tag{26}$$

where the sum on the right is taken over the set J^o of pairs $(\eta,\psi) \in J$ for which $p \nmid m_\eta$.

Let u_k, v_k and w_k be scalars defined for integer $k > 2$ by

$$u_k := \frac{\mathscr{L}_{\rm an}(\chi, k)^{-1}}{c_k}, \qquad v_k := \frac{\mathscr{L}_{\rm an}(\chi^{-1}, k)^{-1}\langle m\rangle^{k-1}}{c_k}, \qquad w_k := \frac{1}{c_k}, \tag{27}$$

where

$$c_k := \mathscr{L}_{\rm an}(\chi, k)^{-1} + \mathscr{L}_{\rm an}(\chi^{-1}, k)^{-1}\langle m\rangle^{k-1} + 1.$$

The vector (u_k, v_k, w_k) is proportional to $(a_k(1, \chi\omega^{1-k}), a_k(\chi, \omega^{1-k}), -1)$. From (26), it follows that the modular form

$$H_k := u_k E_k(1, \chi\omega^{1-k}) + v_k E_k(\chi, \omega^{1-k}) + w_k P_k^o \tag{28}$$

can be written as a linear combination

$$H_k = \begin{pmatrix} \text{A cusp} \\ \text{form} \end{pmatrix} + \sum_{\substack{(\eta,\psi)\in J^o \\ \eta\neq 1,\chi}} w_k a_k(\eta, \psi) E_k(\eta, \psi). \tag{29}$$

In Section 2.5, we introduce a Hecke operator t that satisfies

$$tE_1(1, \chi\omega^0) = E_1(1, \chi\omega^0) \tag{30}$$

and annihilates all the Eisenstein series contributions in (29), so that

$$F_k := tH_k$$

is a cusp form. Equation (30) explains the necessity of calculating the coefficients in (25) instead of applying a Hecke operator t directly to P_k^o—any Hecke operator that annihilates $E_k(1, \chi\omega^{1-k})$ or $E_k(\chi, \omega^{1-k})$ necessarily annihilates their common weight 1 specialization $E_1(1, \chi\omega^0) = E_1(\chi, \omega^0)$.

p-adic interpolation of modular forms

In Section 3 we describe the p-adic interpolation (in the variable k) of the Eisenstein series $E_k(\eta, \psi)$ as well as of the forms P_k, P_k^o, H_k and F_k. For this purpose, the Iwasawa algebra Λ is defined in Section 3.1. In our application, it is most useful to view Λ as a complete subring of the ring $\mathcal{C}(\mathbf{Z}_p, E)$ of continuous E-valued functions on $\mathbf{Z}_p$ equipped with the topology of uniform convergence inherited from the sup norm. The ring Λ is the completion of the $\mathcal{O}_E$-subalgebra of $\mathcal{C}(\mathbf{Z}_p, E)$ generated by the functions $k \mapsto a^k$ as a ranges over $1+p\mathbf{Z}_p$. For each $k \in \mathbf{Z}_p$, write $\nu_k : \Lambda \longrightarrow E$ for the evaluation homomorphism $\nu_k(h) := h(k)$. The algebra Λ is known to be isomorphic to the power series ring $\mathcal{O}_E[\![T]\!]$, and in particular, by the Weierstrass preparation theorem, any element of Λ has finitely many zeroes. If h belongs to the fraction field $\mathcal{F}_\Lambda$ of Λ, it follows that the evaluation $\nu_k(h) \in E$ is defined for all but finitely many $k \in \mathbf{Z}_p$. A Λ-adic modular form of tame level m and character χ is, by definition, a formal q-expansion $\mathscr{G} \in \mathcal{F}_\Lambda \otimes \Lambda[\![q]\!]$ satisfying

$$\nu_k(\mathscr{G}) \text{ belongs to } M_k(mp, \chi\omega^{1-k}) \quad \text{for almost all integers } k > 1.$$

(Here, $\nu_k(\mathscr{G}) \in E[\![q]\!]$ is simply the power series obtained from $\mathscr{G}$ by applying ν_k to its coefficients.) Such a $\mathscr{G}$ is said to be a Λ-*adic cusp form* if $\nu_k(\mathscr{G})$ is a cusp form for all but finitely many $k > 1$. The $\mathcal{F}_\Lambda$-vector spaces of Λ-adic modular forms and cusp forms are denoted $\mathcal{M}(m,\chi)$ and $\mathcal{S}(m,\chi)$, respectively. The ordinary projection e acts on $\mathcal{M}(m,\chi)$ and $\mathcal{S}(m,\chi)$ in a manner compatible with the projections ν_k. The images of e in $\mathcal{M}(m,\chi)$ and $\mathcal{S}(m,\chi)$ are denoted $\mathcal{M}^o(m,\chi)$ and $\mathcal{S}^o(m,\chi)$, respectively. These definitions are all recalled in the context of Hilbert modular forms in Section 3.1.

Basic examples of Λ-adic forms are given by the Λ-adic Eisenstein series $\mathscr{E}(\eta,\psi)$ satisfying

$$\begin{aligned}\nu_k(\mathscr{E}(\eta,\psi)) &= E_k(\eta,\psi\omega^{1-k}) \quad \text{for } k \in \mathbf{Z}^{\geq 2} \\ &= \frac{1}{2}\delta_{\eta=1} L_{S,p}(\psi\omega, 1-k) + \sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n\\ p\nmid d}} \eta\left(\frac{n}{d}\right)\psi(d)\langle d\rangle^{k-1}\Big)q^n.\end{aligned}$$

(By convention, the character $\psi\omega^{1-k}$ is viewed as having modulus divisible by p, even if p does not divide its conductor; in particular $\psi\omega^{1-k}(p) = 0$ for all k.) The existence and basic properties of the Λ-adic Eisenstein series, which are intimately related to those of the p-adic L-functions attached to abelian characters of G_F, are recalled in Section 3.2.

In Section 3.3 we define elements $\mathscr{P} \in \mathcal{M}(m,\chi)$, ordinary forms $\mathscr{P}^o, \mathscr{H} \in \mathcal{M}^o(m,\chi)$, and a cusp form $\mathscr{F} \in \mathcal{S}^o(m,\chi)$ satisfying

$$\nu_k(\mathscr{P}) = P_k, \quad \nu_k(\mathscr{P}^o) = P_k^o, \quad \nu_k(\mathscr{H}) = H_k, \quad \nu_k(\mathscr{F}) = F_k$$

for almost all $k \in \mathbf{Z}^{>2}$. The proof consists essentially in observing that there are elements u, v, and $w \in \mathcal{F}_\Lambda$ such that, for almost all $k > 2$,

$$\nu_k(u) = u_k, \quad \nu_k(v) = v_k, \quad \nu_k(w) = w_k,$$

where u_k, v_k, and w_k are the constants introduced in (27). In Section 3.3, we describe the main properties of the Λ-adic cusp form $\mathscr{F}$, most notably:

1. Let
$$\Lambda_{(1)} := \left\{\frac{f}{g}, \quad \text{with } f, g \in \Lambda \text{ and } g(1) \neq 0\right\}$$
denote the localization of Λ at $\ker \nu_1$. Under the assumption (11) when $\mathfrak{p}$ is the unique prime above p, it can be seen that the elements u, v and w belong to $\Lambda_{(1)}$, and that u even belongs to $\Lambda_{(1)}^\times$. In particular, the cusp form $\mathscr{F}$ belongs to $\Lambda_{(1)} \otimes_\Lambda \Lambda[\![q]\!]$, and hence its weight one specialization is defined. Furthermore, assuming Leopoldt's conjecture for F, we have
$$\nu_1(\mathscr{F}) = E_1(1, \chi\omega^0).$$

2. Let
$$\nu_{1+\varepsilon} : \Lambda_{(1)} \longrightarrow E[\varepsilon]/\varepsilon^2, \qquad \nu_{1+\varepsilon}(f) := f(1) + f'(1)\varepsilon \tag{31}$$
be the natural lift of ν_1 to the ring $\tilde{E} := E[\varepsilon]/\varepsilon^2$ of dual numbers over E. The expression $F_{1+\varepsilon} := \nu_{1+\varepsilon}(\mathscr{F})$ can be thought of as a "cusp form of weight $1+\varepsilon$". Let $u_1 := u(1)$ and

$v_1 = v(1)$ be the values at $k = 1$ of the elements u, v. A direct calculation, explained in Section 3.3, shows that for every prime q,

$$a_q(F_{1+\varepsilon}) = \begin{cases} (1 + v_1\kappa_{\mathrm{cyc}}(q)\varepsilon) + \chi(q)(1 + u_1\kappa_{\mathrm{cyc}}(q)\varepsilon) & \text{if } q \nmid mp, \\ 1 + v_1\kappa_{\mathrm{cyc}}(q)\varepsilon & \text{if } q \mid m, \\ 1 + u_1\mathscr{L}_{\mathrm{an}}(\chi)\varepsilon & \text{if } q = p. \end{cases} \tag{32}$$

An examination of the Fourier coefficients of $F_{1+\varepsilon}$ reveals that it is an eigenform relative to the natural action of the Hecke operators on $\tilde{E}[\![q]\!]$ inherited from the specialization map $\nu_{1+\varepsilon} : \mathcal{M}(m, \chi) \longrightarrow \tilde{E}[\![q]\!]$. In descriptive terms, the Λ-adic cusp form $\mathscr{F}$ is an "eigenform in a first order infinitesimal neighborhood of weight one."

Let $\mathbf{T}$ denote the Λ-algebra generated by the Hecke operators acting on the finite-dimensional $\mathcal{F}_\Lambda$-vector space $\mathcal{S}^o(m, \chi)$. The eigenform $F_{1+\varepsilon}$ gives rise to surjective $\Lambda_{(1)}$-algebra homomorphisms

$$\phi_1 : \mathbf{T} \otimes \Lambda_{(1)} \longrightarrow E, \qquad \phi_{1+\varepsilon} : \mathbf{T} \otimes \Lambda_{(1)} \longrightarrow \tilde{E}$$

sending $t \in \mathbf{T} \otimes \Lambda_{(1)}$ to the associated eigenvalues of t acting on F_1 and $F_{1+\varepsilon}$, respectively. Write $\mathbf{T}_{(1)}$ for the localization of $\mathbf{T} \otimes \Lambda_{(1)}$ at the maximal ideal $\mathfrak{m} := \ker \phi_1$. The homomorphisms ϕ_1 and $\phi_{1+\varepsilon}$ factor through $\mathbf{T}_{(1)}$, and we will often view them as defined on this quotient of $\mathbf{T} \otimes \Lambda_{(1)}$. Likewise, we will write $\mathfrak{m}$ for the maximal ideal of $\mathbf{T}_{(1)}$.

Representations associated to Λ-adic forms

In Section 4 we exploit the homomorphism $\phi_{1+\varepsilon}$ to construct the desired cocycle κ. Let $\mathcal{F}_{(1)}$ denote the total ring of fractions of $\mathbf{T}_{(1)}$:

$$\mathcal{F}_{(1)} := \left\{ \frac{a}{b}, \quad a, b \in \mathbf{T}_{(1)}, \quad b \text{ is not a zero divisor} \right\}.$$

A key ingredient in the construction of κ is the two-dimensional Galois representation

$$\rho : G_F \longrightarrow \mathbf{GL}_2(\mathcal{F}_{(1)}); \qquad \rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$$

attached to the space of ordinary Λ-adic cusp forms. The existence and basic properties of this representation, which were established by Wiles in [16], are recalled in Section 4.1. It is shown in Section 4.1 that ρ can be conjugated so that, for all $\sigma \in G_F$,

$$a(\sigma), d(\sigma) \text{ belong to } \mathbf{T}_{(1)}^\times.$$

Under the assumption that $L_R(\chi, 0) \neq 0$, it is further shown in Section 4.2 that there is a multiple $\tilde{b}$ of the matrix entry b (by some element of $\mathcal{F}_{(1)}$) for which:

1. The function $\sigma \mapsto K(\sigma) := \tilde{b}(\sigma)/d(\sigma)$ takes values in the maximal ideal $\mathfrak{m} \subset \mathbf{T}_{(1)}$.

2. Up to scaling by $E^\times$, the function κ given by the formula

$$\phi_{1+\varepsilon}(K(\sigma)) =: \kappa(\sigma)\varepsilon$$

has the required properties—namely, it is a 1-cocycle representing a class

$$[\kappa] \in H^1_{\mathfrak{p}}(F, E(\chi^{-1}))^{\mathrm{cyc}}$$

that satisfies

$$\mathrm{res}_{\mathfrak{p}}([\kappa]) = -\mathscr{L}_{\mathrm{an}}(\chi)\kappa_{\mathrm{nr}} + \kappa_{\mathrm{cyc}}.$$

1 Duality and the $\mathscr{L}$-invariant

We begin by recalling some notations and conventions regarding characters of F. Fix an ordering of the n real places of F, and let $\mathrm{sgn}_i : \mathbf{R}^\times \to \{\pm 1\}$ denote the associated sign function. For a vector $r \in (\mathbf{Z}/2\mathbf{Z})^n$ write $\mathrm{sgn}(a)^r = \prod_{i=1}^n \mathrm{sgn}_i(a)^{r_i}$.

Let $\mathfrak{b}$ be an integral ideal of F. Let $I_{\mathfrak{b}}$ denote the group of fractional ideals of F that are relatively prime to $\mathfrak{b}$. A *narrow ray class character modulo* $\mathfrak{b}$ is a homomorphism

$$\psi : I_{\mathfrak{b}} \to \overline{\mathbf{Q}}^\times$$

that is trivial on any principal ideal (α) generated by a totally positive element $\alpha \equiv 1 \pmod{\mathfrak{b}}$. The function ψ may be extended to a function on the set of all integral ideals of F by defining $\psi(\mathfrak{m}) = 0$ if $\mathfrak{m}$ is not relatively prime to $\mathfrak{b}$. The character ψ may be viewed as a character modulo $\mathfrak{b}\mathfrak{a}$ for any integral ideal $\mathfrak{a}$. If there exists a narrow ray class character ψ_0 modulo $\mathfrak{c}$ for some proper divisor $\mathfrak{c}$ of $\mathfrak{b}$, with $\psi_0(\mathfrak{m}) = \psi(\mathfrak{m})$ for all $\mathfrak{m}$ relatively prime to $\mathfrak{b}$, then ψ is called an *imprimitive character*. The minimal such divisor of $\mathfrak{b}$ is called the *conductor* of ψ, and ψ_0 is called the associated *primitive character*.

Given a narrow ray class character ψ modulo $\mathfrak{b}$, there exists an $r \in (\mathbf{Z}/2\mathbf{Z})^n$ such that

$$\psi((\alpha)) = \mathrm{sgn}(\alpha)^r \quad \text{for } \alpha \equiv 1 \pmod{\mathfrak{b}}.$$

The vector r is called the *sign* of ψ. The character ψ is said to be *totally even* if $r = (0, 0, \ldots, 0)$ and *totally odd* if $r = (1, 1, \ldots, 1)$.

We place ourselves in the situation of the introduction regarding the character χ and the finite sets R and S of places of the totally real field F. More precisely, we assume that χ is a primitive character of conductor $\mathfrak{n}$ and that S consists exactly of the set of places dividing $\mathfrak{n}p\infty$, while $R = S - \{\mathfrak{p}\}$. Recall that E is a finite extension of $\mathbf{Q}_p$ containing the values of the character χ.

Lemma 1.1. *The following are equivalent:*

1. *The special value $L_R(\chi, 0)$ is non-zero.*

2. *For all places $v \in R$, we have $\chi(v) \neq 1$. (In particular, the character χ is totally odd.)*

3. The vector space U_χ is one-dimensional over E.

Proof. Since χ is non-trivial, the L-series $L_R(\chi^{-1}, s)$ (viewing χ as a complex character) is holomorphic and non-vanishing at $s = 1$. A consideration of the local factors in the functional equation relating $L_R(\chi^{-1}, s)$ and $L_R(\chi, 1-s)$ shows that

$$\mathrm{ord}_{s=0} L_R(\chi, s) = \#\{v \in R \text{ such that } \chi(v) = 1\}. \tag{33}$$

This implies the equivalence of (1) and (2). (See, for example, Prop. 3.4 in Ch. I of [14].) Dirichlet's S-unit theorem implies the equivalence of (2) and (3). (See also Ch. I.4 of [14].) □

Recall that the character χ is said to be of "type S" in the terminology of Greenberg (cf. the first paragraph in [17]) if $H \cap F_\infty = F$, where F_∞ is the cyclotomic $\mathbf{Z}_p$-extension of F. This condition is satisfied, for example, if χ has order prime to p. The following lemma (whose proof is deep, relying on the full force of the main conjecture for totally real fields) essentially disposes of Conjecture 1 in the case where $L_R(\chi, 0) = 0$.

Lemma 1.2. *Suppose that $H \cap F_\infty = F$. If $L_R(\chi, 0) = 0$, then $L'_{S,p}(\chi, 0) = 0$, and therefore Conjecture 1 holds for χ and R.*

Proof. Theorem 1.2 of [17] implies that

$$\mathrm{ord}_{s=0} L_{S,p}(\chi\omega, s) = \mathrm{ord}_{s=0} f_\chi, \tag{34}$$

where f_χ corresponds to the characteristic power series of the $\Lambda = \mathbf{Z}_p[\![\mathrm{Gal}(H_\infty/H)]\!]$-module $X_\chi \subset \mathrm{Gal}(L_\infty/H_\infty)$ defined on p. 409 of [15]. (Here H_∞ denotes the cyclotomic $\mathbf{Z}_p$-extension of H and L_∞ is the maximal unramified abelian pro-p-extension of H_∞.) It is known by an explicit construction (cf. the statement at the top of p. 410 of [15], or [1], [5]) that

$$\#\{v \in S \text{ such that } \chi(v) = 1\} \le \mathrm{ord}_{s=0} f_\chi. \tag{35}$$

Since $\chi(\mathfrak{p}) = 1$ and $\mathfrak{p} \in S$, equations (33) and (35) combine with (34) to give

$$\mathrm{ord}_{s=0} L_{S,p}(\chi\omega, s) \ge \mathrm{ord}_{s=0} L_R(\chi, s) + 1.$$

In particular, $L'_{S,p}(\chi\omega, 0) = 0$ when $L_R(\chi, 0) = 0$. □

Thanks to Lemma 1.2, we assume for the rest of this article χ satisfies the equivalent assumptions of Lemma 1.1.

Let $E(1)(\chi)$ and $E(\chi^{-1})$ be the continuous one-dimensional representations of G_F defined in the introduction, following equation (14). We will now study certain Galois cohomology groups (both local and global) associated to these p-adic representations.

If K is a field and M is a finite G_K-module, we denote by $H^i(K, M)$ the Galois cohomology group of *continuous* i-cocycles on G_K with values in M, modulo i-coboundaries. If $\mathcal{O}_E$ denotes the rings of integers of E and π is a uniformizing element of $\mathcal{O}_E$, then recall the definitions:

$$\begin{aligned} H^1(K, \mathcal{O}_E(\chi^{-1})) &:= \lim_{\leftarrow, n} H^1(K, \mathcal{O}_E/\pi^n(\chi^{-1})), \\ H^1(K, E(\chi^{-1})) &:= H^1(K, \mathcal{O}_E(\chi^{-1})) \otimes_{\mathcal{O}_E} E. \end{aligned}$$

1.1 Local cohomology groups

Let v be any place of F, and let G_v and $I_v \subset G_v$ denote a choice of decomposition and inertia group at v in G_F. Given any finite G_F-module M, the inflation-restriction sequence attached to I_v yields the exact sequence

$$0 \longrightarrow H^1(G_v/I_v, M^{I_v}) \longrightarrow H^1(F_v, M) \xrightarrow{\mathrm{res}_{I_v}} H^1(I_v, M)^{G_v/I_v}. \tag{36}$$

A class in $H^1(F, M)$ is said to be *unramified* at v if its restriction to $H^1(F_v, M)$ lies in the kernel of res_{I_v}.

Recall that local Tate duality gives rise to a perfect pairing

$$H^1(F_v, \mathbf{Z}/p^n\mathbf{Z}) \times H^1(F_v, \mu_{p^n}) \longrightarrow H^2(F_v, \mu_{p^n}) = \mathbf{Z}/p^n\mathbf{Z}.$$

After tensoring this with $\mathcal{O}_E/\pi^n$, twisting by χ^{-1}, and passing to the limit as $n \to \infty$, one obtains perfect $\mathcal{O}_E$-linear and E-linear pairings

$$\langle\ ,\ \rangle_v \ :\ H^1(F_v, \mathcal{O}_E/\pi^n(\chi^{-1})) \times H^1(F_v, \mathcal{O}_E/\pi^n(1)(\chi)) \longrightarrow \mathcal{O}_E/\pi^n, \tag{37}$$

$$\langle\ ,\ \rangle_v \ :\ H^1(F_v, E(\chi^{-1})) \times H^1(F_v, E(1)(\chi)) \longrightarrow E. \tag{38}$$

If $\chi(v) \neq 1$, then $H^1(G_v/I_v, \mathcal{O}_E/\pi^n(\chi^{-1})^{I_v}) \cong (\mathcal{O}_E/\pi^n(\chi^{-1})^{I_v})/(\chi^{-1}(v) - 1)$ has cardinality bounded independently of n, and there are no unramified classes in $H^1(F_v, E(\chi^{-1}))$. If $\chi(v) = 1$, we have:

1. The group $H^1(F_v, E(\chi^{-1})) = H^1(F_v, E) = \mathrm{Hom}_{\mathrm{cts}}(G_v, E)$ contains a distinguished unramified class: the unique homomorphism

$$\kappa_{\mathrm{nr}} \in \mathrm{Hom}(\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v), \mathcal{O}_E)$$

   sending the Frobenius element Frob_v at v to 1.

2. If v divides p, then the restriction to G_{F_v} of the p-adic logarithm of the cyclotomic character

$$\kappa_{\mathrm{cyc}} := \log_p(\epsilon_{\mathrm{cyc}}) \in \mathrm{Hom}(G_F, E) = H^1(F, E)$$

   gives a ramified element of $H^1(F_v, E)$. The elements κ_{nr} and κ_{cyc} will sometimes be referred to as the *unramified* and *cyclotomic* cocycles respectively.

3. For each positive integer n, the connecting homomorphism of Kummer Theory yields an isomorphism

$$\delta_{v,n} : F_v^\times \otimes \mathbf{Z}/p^n\mathbf{Z} \longrightarrow H^1(F_v, \mathbf{Z}/p^n\mathbf{Z}(1))$$

   As n varies, the maps $\delta_{v,n}$ are compatible with the natural projections on both sides. Passing to the limit with n, and then tensoring with E, we obtain an isomorphism

$$\delta_v : F_v^\times \hat{\otimes} E \longrightarrow H^1(F_v, E(1)), \tag{39}$$

   where

$$F_v^\times \hat{\otimes} E := \left(\lim_{\leftarrow, n} F_v^\times \otimes \mathbf{Z}/p^n\mathbf{Z} \right) \otimes_{\mathbf{Z}_p} E.$$

4. For each $u \in F_v^\times \hat{\otimes} E$, we have

$$\langle \kappa_{\rm nr}, \delta_v(u)\rangle_v = -\operatorname{ord}_v(u), \qquad \langle \kappa_{\rm cyc}, \delta_v(u)\rangle_v = \mathbf{L}_v(u), \tag{40}$$

where ord_v and $\mathbf{L}_v$ are the homomorphisms $F_v^\times \hat{\otimes} E \longrightarrow E$ defined as in (5) and (6) of the introduction. The equations in (40) are direct consequences of the reciprocity law of local class field theory. In particular, the subspace of $H^1(F_v, E(1))$ that is orthogonal to $\kappa_{\rm nr}$ under the local Tate pairing is equal to $\delta_v(\mathcal{O}_{F_v}^\times \hat{\otimes} E)$.

Lemma 1.3. *Let v be any place of F. The dimension of $H^1(F_v, E(1)(\chi))$ is given in the following table:*

	$\chi(v) = 1$	$\chi(v) \neq 1$
$v \nmid p\infty$	1	0
$v \mid \infty$	0	0
$v \mid p$	$[F_v : \mathbf{Q}_p] + 1$	$[F_v : \mathbf{Q}_p]$.

The same is true for $H^1(F_v, E(\chi^{-1}))$.

Proof. For v infinite, the lemma is clear. For v finite, Tate's local Euler characteristic formula (see the "Corank Lemma" of [7, Chapter 2]) and Tate local duality yield

$$\begin{aligned} &\dim_E H^1(F_v, E(1)(\chi)) \\ &\quad = \dim_E H^0(F_v, E(1)(\chi)) + \dim_E H^2(F_v, E(1)(\chi)) + \begin{cases} 0 & v \nmid p \\ [F_v : \mathbf{Q}_p] & v \mid p \end{cases} \\ &\quad = \dim_E H^0(F_v, E(\chi^{-1})) + \begin{cases} 0 & v \nmid p \\ [F_v : \mathbf{Q}_p] & v \mid p \end{cases}, \end{aligned}$$

which explains the remaining values in the table. ☐

1.2 Global cohomology groups

We now define certain subgroups of the global cohomology groups $H^1(F, E(1)(\chi))$ and $H^1(F, E(\chi^{-1}))$ by imposing appropriate local conditions.

Recall that

$$H^1_{\rm nr}(F_v, E(\chi^{-1})) \simeq H^1(G_v/I_v, E(\chi^{-1})^{I_v}) = \begin{cases} E \cdot \kappa_{\rm nr} & \text{if } \chi(v) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

denotes the subgroup of unramified cohomology classes. We denote by $H^1_{\rm nr}(F_v, E(\chi)(1))$ the orthogonal complement of $H^1_{\rm nr}(F_v, E(\chi^{-1}))$ under the local Tate pairing, so

$$H^1_{\rm nr}(F_v, E(\chi)(1)) \simeq \begin{cases} \mathcal{O}_{F_v}^\times \hat{\otimes} E & \text{if } \chi(v) = 1; \\ H^1(F_v, E(\chi)(1)) & \text{otherwise.} \end{cases}$$

Let

$$H^1_{[\mathfrak{p}]}(F, E(\chi^{-1})) \subset H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$$

denote the subgroups of $H^1(F, E(\chi^{-1}))$ consisting of classes whose restriction to G_{F_v} belong to $H^1_{\mathrm{nr}}(F_v, E(\chi^{-1}))$ for all $v \neq \mathfrak{p}$, and which are trivial/arbitrary, respectively, at the prime $\mathfrak{p}$. Similarly, we denote by

$$H^1_{[\mathfrak{p}]}(F, E(1)(\chi)) \subset H^1_{\mathfrak{p}}(F, E(1)(\chi))$$

the subgroups of classes in $H^1(F, E(1)(\chi))$ whose restrictions lie in $H^1_{\mathrm{nr}}(F_v, E(\chi)(1))$ for all $v \neq \mathfrak{p}$, and which are trivial/arbitrary, respectively, at the prime $\mathfrak{p}$.

Recall the group U_χ of Gross-Stark units that was defined in equation (4) of the introduction.

Proposition 1.4. *The natural map*

$$\delta : U_\chi \longrightarrow H^1_{\mathfrak{p}}(F, E(1)(\chi))$$

induced by the connecting homomorphism of Kummer theory is an isomorphism. In particular,

$$\dim_E H^1_{\mathfrak{p}}(F, E(1)(\chi)) = 1.$$

Proof. If $G = \mathrm{Gal}(H/F)$, then the restriction map

$$H^1(F, E(1)(\chi)) \xrightarrow{\mathrm{res}} H^1(H, E(1)(\chi))^G \tag{41}$$

induces an isomorphism. Indeed, the kernel and the cokernel of this map are given by $H^i(G, E(1)(\chi)^{G_H})$ for $i = 1, 2$ where $G_H = \mathrm{Gal}(\bar{F}/H)$, and $E(1)(\chi)^{G_H} = 0$. Thus

$$H^1(F, E(1)(\chi)) \xrightarrow{\simeq} H^1(H, E(1)(\chi))^G = H^1(H, E(1))^{\chi^{-1}} \simeq (H^\times \hat{\otimes} E)^{\chi^{-1}}, \tag{42}$$

where the superscript of χ^{-1} affixed to an $E[G]$-module denotes the submodule on which G acts via χ^{-1}. The last isomorphism in (42) arises from (39) with F_v replaced by H.

Similarly, in the local situation, we have

$$H^1(F_v, E(1)(\chi)) \simeq (H_w^\times \hat{\otimes} E)^{\chi^{-1}}. \tag{43}$$

where w is some prime of H over v. The result now follows from the fact that the group $H^1_{\mathrm{nr}}(F_v, E(1)(\chi))$ corresponds to $(\mathcal{O}_{H_w}^\times \hat{\otimes} E)^{\chi^{-1}}$ under this identification. □

If W is any subspace of the local cohomology group $H^1(F_{\mathfrak{p}}, E(\chi^{-1})) = \mathrm{Hom}(G_{\mathfrak{p}}, E)$, define

$$H^1_{\mathfrak{p},W}(F, E(\chi^{-1})) \subset H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$$

to be the subspace consisting of classes whose image under $\mathrm{res}_{\mathfrak{p}}$ belongs to W.

The following lemma can be viewed as the global counterpart of Lemma 1.3.

Lemma 1.5. *Let χ and $\mathfrak{p}$ satisfy $\chi(\mathfrak{p}) = 1$. Then*

$$\dim_E H^1_{\mathfrak{p}}(F, E(\chi^{-1})) = [F_{\mathfrak{p}} : \mathbf{Q}_p].$$

More generally, if $W \subset H^1(F_{\mathfrak{p}}, E)$ is any subspace containing the unramified cocycle κ_{nr}, then

$$\dim_E H^1_{\mathfrak{p},W}(F, E(\chi^{-1})) = \dim_E W - 1.$$

Proof. The Poitou-Tate exact sequence in Galois cohomology, applied to the finite modules $\mathcal{O}_E/\pi^n(\chi^{-1})$, gives rise to the exact sequence

$$\begin{aligned} 0 \longrightarrow H^1_{[\mathfrak{p}]}(F, \mathcal{O}_E/\pi^n(\chi^{-1})) \longrightarrow H^1_{\mathfrak{p}}(F, \mathcal{O}_E/\pi^n(\chi^{-1})) \longrightarrow H^1(F_{\mathfrak{p}}, \mathcal{O}_E/\pi^n(\chi^{-1})) \\ \longrightarrow H^1_{\mathfrak{p}}(F, \mathcal{O}_E/\pi^n(1)(\chi))^\vee, \end{aligned} \tag{44}$$

where the last map arises from the local Tate pairing. Now we note that the module $H^1_{[\mathfrak{p}]}(F, \mathcal{O}_E/\pi^n(\chi^{-1}))$ maps (with kernel bounded independently of n) to the group of homomorphisms from G_H to $\mathcal{O}_E/\pi^n$ that are everywhere unramified. Hence $H^1_{[\mathfrak{p}]}(F, \mathcal{O}_E/\pi^n(\chi^{-1}))$ has bounded cardinality as $n \to \infty$. Passing to the limit with n and tensoring with E, we obtain the exact sequence

$$0 \longrightarrow H^1_{\mathfrak{p}}(F, E(\chi^{-1})) \longrightarrow H^1(F_{\mathfrak{p}}, E(\chi^{-1})) \longrightarrow H^1_{\mathfrak{p}}(F, E(1)(\chi))^\vee. \tag{45}$$

We now observe that the element κ_{nr} maps to a non-zero element of the one-dimensional vector space $H^1_{\mathfrak{p}}(F, E(\chi)(1))^\vee$, by (40) and Proposition 1.4. Hence the last arrow in (45) is surjective, and the same is true for the exact sequence

$$0 \longrightarrow H^1_{\mathfrak{p},W}(F, E(\chi^{-1})) \longrightarrow W \longrightarrow H^1_{\mathfrak{p}}(F, E(1)(\chi))^\vee,$$

for any $W \subset H^1(F_{\mathfrak{p}}, E)$ containing κ_{nr}. The lemma now follows from Proposition 1.4. ☐

1.3 A formula for the $\mathscr{L}$-invariant

Let W_{cyc} be the subspace of $H^1(F_{\mathfrak{p}}, E)$ spanned by the unramified and cyclotomic cocycles κ_{nr} and κ_{cyc}. Write

$$H^1_{\mathfrak{p},\mathrm{cyc}}(F, E(\chi^{-1})) := H^1_{\mathfrak{p},W_{\mathrm{cyc}}}(F, E(\chi^{-1})).$$

By Lemma 1.5, this space is one-dimensional over E. If κ is any non-zero element of this space, we may therefore write

$$\mathrm{res}_{\mathfrak{p}}\, \kappa = x \cdot \kappa_{\mathrm{nr}} + y \cdot \kappa_{\mathrm{cyc}}, \tag{46}$$

for some $x, y \in E$, and the ratio x/y does not depend on the choice of κ. (Note that $y \neq 0$, or else (46) contradicts Lemma 1.3 when W is the 1-dimensional space spanned by κ^{nr}.)

Proposition 1.6. *Let $\mathscr{L}(\chi)$ be the $\mathscr{L}$-invariant introduced in equation (7) of the introduction. Then*

$$\mathscr{L}(\chi) = -x/y.$$

Proof. The global reciprocity law of class field theory applied to $\delta(u_\chi) \in H^1_{\mathfrak{p}}(F, E(\chi)(1))$ with $u_\chi \in U_\chi$ and $\kappa \in H^1_{\mathfrak{p},\mathrm{cyc}}(F, E(\chi^{-1}))$ implies that

$$\sum_v \langle \mathrm{res}_v\,\kappa, \delta_v(u_\chi)\rangle_v = 0,$$

where the sum is taken over all places v of F. By definition, all the terms in this sum vanish except the one corresponding to $v = \mathfrak{p}$. It follows that

$$\langle \mathrm{res}_{\mathfrak{p}}\,\kappa, \delta_{\mathfrak{p}}(u_\chi)\rangle_{\mathfrak{p}} = 0.$$

Combining the expression (46) for $\mathrm{res}_{\mathfrak{p}}\,\kappa$ with (40) yields the proposition. □

2 Hilbert modular forms

2.1 Definitions

We briefly describe the basic definitions of classical Hilbert modular forms over a totally real field F. Recall that we have chosen an ordering of the n real embeddings of F. For an element $z \in \mathbf{C}^n$, an integer k, and elements $a, b \in F$, define

$$(az+b)^k := \prod_{i=1}^{n} (a_i z_i + b_i)^k \in \mathbf{C},$$

where a_i, b_i denote the images of a, b, respectively, under the ith real embedding of F.

Let ψ be a narrow ray class character modulo $\mathfrak{b}$ with sign $r \in (\mathbf{Z}/2\mathbf{Z})^n$. Let $\alpha \in \mathcal{O}_F$ be relatively prime to $\mathfrak{b}$. The map $\alpha \mapsto \mathrm{sgn}(\alpha)^r \psi((\alpha))$ defines a character

$$\psi_f : (\mathcal{O}_F/\mathfrak{b})^\times \to \overline{\mathbf{Q}}^\times$$

associated to ψ.

Let k be a positive integer. In [12, Page 649], Shimura defines a space $M_k(\mathfrak{b}, \psi)$ of Hilbert modular forms of level $\mathfrak{b}$ and character ψ. Let h denote the size of the narrow class group $\mathrm{Cl}^+(F)$. An element $f \in M_k(\mathfrak{b}, \psi)$ is an h-tuple of holomorphic functions

$$f_\lambda : \mathcal{H}^n \to \mathbf{C}$$

for $\lambda \in \mathrm{Cl}^+(F)$ satisfying certain modularity properties that we now describe (see (2.5a) and (2.15a-c) of [12]).

For each $\lambda \in \mathrm{Cl}^+(F)$, choose a representative fractional ideal $\mathfrak{t}_\lambda$. Let $\mathbf{GL}_2^+(F)$ denote the group of 2×2 invertible matrices over F that have positive determinant at each real place of F. Define

$$\Gamma_\lambda := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2^+(F) : \quad a, d \in \mathcal{O}_F, \quad b \in \mathfrak{t}_\lambda^{-1}\mathfrak{d}^{-1}, \quad c \in \mathfrak{b}\mathfrak{t}_\lambda\mathfrak{d}, \quad ad - bc \in \mathcal{O}_F^\times \right\},$$

where $\mathfrak{d}$ denotes the different of F. Each function f_λ satisfies

$$f_\lambda|_\gamma = \psi_f(a) f_\lambda \tag{47}$$

for all $\gamma \in \Gamma_\lambda$, where

$$f_\lambda|_\gamma(z) := \det(\gamma)^{k/2}(cz+d)^{-k} f_\lambda(\gamma z). \tag{48}$$

In (48), the positive square root is taken in

$$\det(\gamma)^{k/2} := \prod_{i=1}^{n} \det(\gamma_i)^{k/2}$$

and

$$\gamma z := \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_n z_n + b_n}{c_n z_n + d_n} \right).$$

In [12, Page 648], Shimura defines a Hecke operator $S(\mathfrak{m})$ for each integral ideal $\mathfrak{m}$ relatively prime to $\mathfrak{b}$ on the space of h-tuples $f = (f_\lambda)$ satisfying (47). We do not recall the definition of $S(\mathfrak{m})$ here. The space $M_k(\mathfrak{b}, \psi)$ is defined as the set of f such that

$$S(\mathfrak{m}) f = \psi(\mathfrak{m}) f, \qquad \text{for all } \mathfrak{m} \text{ with } (\mathfrak{m}, \mathfrak{b}) = 1.$$

The modularity property (47) implies that f_λ has a Fourier expansion

$$f_\lambda(z) = a_\lambda(0) + \sum_{\substack{b \in \mathfrak{t}_\lambda \\ b \gg 0}} a_\lambda(b) e_F(bz),$$

where $e_F(x) = \exp(2\pi i \cdot \mathrm{Tr}_{F/\mathbf{Q}}(x))$. We call the coefficients $a_\lambda(b)$ the *unnormalized* Fourier coefficients of f, and define also the *normalized* Fourier coefficients $c(\mathfrak{m}, f)$ and $c_\lambda(0, f)$ of f as follows. Any non-zero integral ideal $\mathfrak{m}$ may be written $\mathfrak{m} = (b)\mathfrak{t}_\lambda^{-1}$ with b totally positive (and $b \in \mathfrak{t}_\lambda$) for a unique $\lambda \in \mathrm{Cl}^+(F)$. Define

$$c(\mathfrak{m}, f) := a_\lambda(b) \, \mathrm{N}(\mathfrak{t}_\lambda)^{-k/2}. \tag{49}$$

The right side of (49) is easily seen to depend only on $\mathfrak{m}$ and not on the choice of b since $f_\lambda(\epsilon z) \, \mathrm{N}\epsilon^{k/2} = f_\lambda(z)$ for every totally positive unit ϵ of F. Similarly we define

$$c_\lambda(0, f) := a_\lambda(0) \, \mathrm{N}(\mathfrak{t}_\lambda)^{-k/2}$$

for each $\lambda \in \mathrm{Cl}^+(F)$. Note that our normalized Fourier coefficients $c(\mathfrak{m}, f)$ are denoted $C(\mathfrak{m}, f)$ in [12]; we have chosen to remain consistent with the notation of [17].

If for each $\gamma \in \mathbf{GL}_2^+(F)$ and $\lambda \in \mathrm{Cl}^+(F)$, the function $f_\lambda|_\gamma$ has constant term equal to 0, then f is called a cusp form. The space of cusp forms of weight k, level $\mathfrak{b}$ and character ψ is denoted $S_k(\mathfrak{b}, \psi)$.

2.2 Eisenstein Series

The most basic examples of Hilbert modular forms arise from Eisenstein series attached to pairs of narrow ideal class characters of F.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be integral ideals of F, and let η and ψ be (possibly imprimitive) E-valued characters of the narrow ideal class group modulo $\mathfrak{a}$ and $\mathfrak{b}$, respectively, with associated signs $q, r \in (\mathbf{Z}/2\mathbf{Z})^n$. Let $k \geq 1$ be an integer satisfying

$$q + r \equiv (k, k, \ldots, k) \pmod{2\mathbf{Z}^n}.$$

We view the characters $\eta\psi$ and $\eta\psi^{-1}$ as having modulus $\mathfrak{ab}$. Here and elsewhere, an expression like $L(\psi\eta^{-1}, 1-k)$ without a subscript on the L is assumed to be taken with the set S in equation (1) equal to the set of archimedean primes and those dividing the modulus of the given character.

Proposition 2.1. *For all $k \geq 1$, there exists an element $E_k(\eta, \psi) \in M_k(\mathfrak{ab}, \eta\psi)$ such that*

$$c(\mathfrak{m}, E_k(\eta, \psi)) = \sum_{\mathfrak{r} \mid \mathfrak{m}} \eta\left(\frac{\mathfrak{m}}{\mathfrak{r}}\right) \psi(\mathfrak{r}) \, \mathrm{N}\mathfrak{r}^{k-1}, \tag{50}$$

for all non-zero ideals $\mathfrak{m}$ of $\mathcal{O}_F$. When $k > 1$, we have

$$c_\lambda(0, E_k(\eta, \psi)) = \begin{cases} 2^{-n}\eta^{-1}(\mathfrak{t}_\lambda) L(\psi\eta^{-1}, 1-k) & \text{if } \mathfrak{a} = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{51}$$

When $k = 1$, we have $E_1(\eta, \psi) = E_1(\psi, \eta)$, and

$$c_\lambda(0, E_1(\eta, \psi)) = 2^{-n} \cdot \begin{cases} \eta^{-1}(\mathfrak{t}_\lambda) L(\psi\eta^{-1}, 0) & \text{if } \mathfrak{a} = 1 \text{ and } \mathfrak{b} \neq 1, \\ \psi^{-1}(\mathfrak{t}_\lambda) L(\eta\psi^{-1}, 0) & \text{if } \mathfrak{b} = 1 \text{ and } \mathfrak{a} \neq 1, \\ \eta^{-1}(\mathfrak{t}_\lambda) L(\psi\eta^{-1}, 0) + \psi^{-1}(\mathfrak{t}_\lambda) L(\eta\psi^{-1}, 0) & \text{if } \mathfrak{a} = \mathfrak{b} = 1, \\ 0 & \text{if } \mathfrak{a} \neq 1 \text{ and } \mathfrak{b} \neq 1. \end{cases} \tag{52}$$

Remark 2.2. This proposition is well-known to the experts, and follows easily, for example, from Katz's computation of the q-expansion of Eisenstein series in Chapter III of [9]. Since our notations differ somewhat from those of [9], we have supplied the details of the proof of formulae (51) and (52) for the sake of being self-contained. Furthermore, some of the objects that are introduced in our proof of (51) and (52) will be used in later calculations. But the reader willing to take Proposition 2.1 on faith may wish to skip the proof on a first reading; in fact, in the remainder of Section 2.2 we assemble computations of constant terms of Eisenstein series at various cusps, and we recommend that the reader continue on to Section 2.3 and refer back to Section 2.2 as necessary.

Proof of Proposition 2.1. The existence of the Eisenstein series $E_k(\eta, \psi)$ satisfying (50) is given by [12, Proposition 3.4]. We recall the definition of $E_k(\eta, \psi)$ given there. Let

$$U = \{u \in \mathcal{O}_F^\times : \mathrm{N}u^k = 1, u \equiv 1 \pmod{\mathfrak{ab}}\}.$$

For $k \geq 1$, we set $E_k(\eta, \psi) = (f_\lambda)$, where $f_\lambda(z)$ is defined via Hecke's trick as follows. For $z \in \mathcal{H}^n$ and $s \in \mathbf{C}$ with $\mathrm{Re}(2s+k) > 2$, define

$$f_\lambda(z,s) := C \cdot \frac{\mathrm{N}(\mathfrak{t}_\lambda)^{-k/2}}{\mathrm{N}\mathfrak{b}} \sum_{\mathfrak{r} \in \mathrm{Cl}(F)} \mathrm{N}\mathfrak{r}^k \sum_{\substack{a \in \mathfrak{r} \\ b \in \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{r} \\ (a,b) \bmod U,\ (a,b) \neq (0,0)}} \frac{\mathrm{sgn}(a)^q \eta(a\mathfrak{r}^{-1})}{(az+b)^k |az+b|^{2s}} \times \sum_{c \in \mathfrak{t}_\lambda \mathfrak{r}^{-1}/\mathfrak{b}\mathfrak{t}_\lambda\mathfrak{r}^{-1}} \mathrm{sgn}(c)^r \psi(c\mathfrak{t}_\lambda^{-1}\mathfrak{r}) e_F(-bc) \tag{53}$$

where

$$C := \frac{\sqrt{d_F} \cdot \Gamma(k)^n}{[\mathcal{O}_F^\times : U]\, \mathrm{N}(\mathfrak{d})(-2\pi i)^{kn}}. \tag{54}$$

The sums in (53) run over representatives $\mathfrak{r}$ for the wide class group $\mathrm{Cl}(F)$, representatives (a, b) for the non-zero elements of the product $\mathfrak{r} \times \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{r}$ modulo the action of U (which acts by diagonal multiplication on the two factors), and representatives c for $\mathfrak{t}_\lambda\mathfrak{r}^{-1}/\mathfrak{b}\mathfrak{t}_\lambda\mathfrak{r}^{-1}$. The term $\mathrm{sgn}(a)^q\eta(a\mathfrak{r}^{-1})$ in (53) for $a = 0$ should be interpreted as being equal to 0 if $\mathfrak{a} \neq 1$ and equal to $\eta(\mathfrak{r}^{-1})$ if $\mathfrak{a} = 1$; a similar interpretation holds for $c = 0$. Let us now assume that ψ is a primitive character, i.e. that the conductor of ψ is $\mathfrak{b}$. From [12, (3.11)], we find that the last sum in (53) may be evaluated as follows:

$$\sum_{c \in \mathfrak{t}_\lambda \mathfrak{r}^{-1}/\mathfrak{b}\mathfrak{t}_\lambda\mathfrak{r}^{-1}} \mathrm{sgn}(c)^r \psi(c\mathfrak{t}_\lambda^{-1}\mathfrak{r}) e_F(-bc) = \mathrm{sgn}(-b)^r \psi^{-1}(-b\mathfrak{d}\mathfrak{b}\mathfrak{t}_\lambda\mathfrak{r}^{-1})\tau(\psi). \tag{55}$$

Here $\tau(\psi)$ is the Gauss sum attached to ψ, defined by

$$\tau(\psi) := \sum_{x \in \mathfrak{b}^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}} \mathrm{sgn}(x)^r \psi(x\mathfrak{b}\mathfrak{d}) e_F(x). \tag{56}$$

When $\mathfrak{b} = 1$, we interpret (56) as $\tau(\psi) = \psi(\mathfrak{d})$.

For fixed z, the function $f_\lambda(z,s)$ has a meromorphic continuation in s to the entire complex plane. The function $f_\lambda(z)$ is defined as $f_\lambda(z, 0)$. Note that if $k > 2$ the Hecke regularization process is not necessary; the series in (53) with $s = 0$ converges and yields the definition of $f_\lambda(z)$.

Suppose that $k > 1$. By [12, (3.7)], the unnormalized constant term $a_\lambda(0)$ of $f_\lambda(z)$ is equal to 0 if $\mathfrak{a} \neq 1$, and is equal to the value of

$$C \cdot \frac{\mathrm{N}(\mathfrak{t}_\lambda)^{-k/2}\tau(\psi)}{\mathrm{N}\mathfrak{b}} \sum_{\mathfrak{r} \in \mathrm{Cl}(F)} \mathrm{N}\mathfrak{r}^k \sum_{\substack{b \in \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{r}/U \\ b \neq 0}} \frac{\eta(\mathfrak{r}^{-1})\, \mathrm{sgn}(-b)^r \psi^{-1}(-b\mathfrak{d}\mathfrak{b}\mathfrak{t}_\lambda\mathfrak{r}^{-1})}{\mathrm{N}b^k |\mathrm{N}b|^{2s}} \tag{57}$$

at $s = 0$ if $\mathfrak{a} = 1$. Note that this is the value obtained by sending each $z_i \to \infty i$ in (53).

The map

$$(\mathfrak{r}, b) \mapsto \mathfrak{f} = (b)\mathfrak{d}\mathfrak{b}\mathfrak{t}_\lambda\mathfrak{r}^{-1}$$

is a surjective $[\mathcal{O}_F^\times : U]$-to-1 map from the set of pairs $(\mathfrak{r}, b)$ in (57) to the set of non-zero integral ideals of F. We therefore find that $a_\lambda(0)$ is the value at $s = 0$ of the function

$$\frac{\sqrt{d_F}\,\mathrm{N}(\mathfrak{t}_\lambda)^{k/2}\,\mathrm{N}(\mathfrak{d}\mathfrak{b})^{k-1}\Gamma(k)^n\tau(\psi)}{\eta(\mathfrak{d}\mathfrak{b}\mathfrak{t}_\lambda)(2\pi i)^{kn}}\sum_{\mathfrak{f}\subset\mathcal{O}_F}\frac{\eta\psi^{-1}(\mathfrak{f})}{\mathrm{N}\mathfrak{f}^k\,\mathrm{N}(\mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{r}\mathfrak{f})^{2s}}, \tag{58}$$

where $\mathfrak{r}$ is the chosen representative of $\mathrm{Cl}(F)$ equivalent to $\mathfrak{d}\mathfrak{b}\mathfrak{t}_\lambda\mathfrak{f}^{-1}$. The value at $s = 0$ of the sum in (58) is evidently $L(\eta\psi^{-1}, k)$. Since $\tau(\psi\eta^{-1}) = \tau(\psi)\eta^{-1}(\mathfrak{d}\mathfrak{b})$, the functional equation for Hecke L-series (see [10, Theorem 3.3.1]) yields

$$a_\lambda(0) = \frac{\mathrm{N}(\mathfrak{t}_\lambda)^{k/2}\eta^{-1}(\mathfrak{t}_\lambda)}{2^n}L(\psi\eta^{-1}, 1-k)$$

as desired.

When $k = 1$, [12, (3.7)] shows that the formula (57) requires an extra term equal to

$$\frac{\mathrm{N}(\mathfrak{t}_\lambda)^{1/2}\tau(\psi)}{2^n[\mathcal{O}_F^\times : U]\,\mathrm{N}\mathfrak{b}}\sum_{\substack{\mathfrak{r}\in\mathrm{Cl}(F)\\ t\in\mathfrak{b}^{-1}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{r}/\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{r}}}\mathrm{sgn}(t)^r\psi^{-1}(t\mathfrak{d}\mathfrak{b}\mathfrak{t}_\lambda\mathfrak{r}^{-1})\sum_{\substack{b\in\mathfrak{r}/U\\ b\neq 0}}\mathrm{sgn}(b)^q\eta(b\mathfrak{r}^{-1})\cdot\mathrm{sgn}(\mathrm{N}b)|\mathrm{N}b|^{-2s}. \tag{59}$$

The value of (59) at $s = 0$ is easily seen to equal 0 if $\mathfrak{b} \neq 1$, and to equal

$$\frac{\mathrm{N}(\mathfrak{t}_\lambda)^{1/2}\psi^{-1}(\mathfrak{t}_\lambda)}{2^n}L(\eta\psi^{-1}, 0)$$

if $\mathfrak{b} = 1$. This completes the proof when ψ is primitive.

If ψ is imprimitive with conductor $\mathfrak{b}_0 \mid \mathfrak{b}$, let ψ_0 denote the associated primitive character. Raising the modulus of ψ_0 at a prime already dividing $\mathfrak{b}_0$ affects neither $L(\eta\psi_0^{-1}, s)$ nor $E_k(\eta, \psi_0)$. Therefore it suffices to consider the case when the modulus of ψ_0 is raised at a prime $\mathfrak{q} \nmid \mathfrak{b}_0$, that is, when $\mathfrak{b} = \mathfrak{b}_0\mathfrak{q}$. Then $L(\eta\psi^{-1}, s)$ is obtained from $L(\eta\psi_0^{-1}, s)$ by removing the Euler factor at $\mathfrak{q}$. Meanwhile, $E_k(\eta, \psi)$ is obtained from $E_k(\eta, \psi_0)$ as follows. For every modular form $f \in M_k(\mathfrak{b}_0, \psi_0)$, there exists a modular form $f|\mathfrak{q} \in M_k(\mathfrak{b}, \psi)$ such that for nonzero integral ideals $\mathfrak{m}$ we have

$$c(\mathfrak{m}, f|\mathfrak{q}) = \begin{cases} c(\mathfrak{m}/\mathfrak{q}, f) & \mathfrak{q} \mid \mathfrak{m} \\ 0 & \mathfrak{q} \nmid \mathfrak{m}, \end{cases}$$

and for $\lambda \in \mathrm{Cl}^+(F)$ we have

$$c_\lambda(0, f|\mathfrak{q}) = c_{\lambda\mathfrak{q}}(0, f)$$

(see [12, Proposition 2.3]). Then it is easy to verify from (50) that

$$E_k(\eta, \psi) = E_k(\eta, \psi_0) - \psi_0(\mathfrak{q})\,\mathrm{N}\mathfrak{q}^{k-1}E_k(\eta, \psi_0)|\mathfrak{q}.$$

Since

$$\left(\eta^{-1}(\mathfrak{t}_\lambda) - \psi_0(\mathfrak{q})\,\mathrm{N}\mathfrak{q}^{k-1}\eta^{-1}(\mathfrak{t}_{\lambda\mathfrak{q}})\right)L(\psi_0\eta^{-1}, 1-k) = \eta^{-1}(\mathfrak{t}_\lambda)L(\psi\eta^{-1}, 1-k)$$

and

$$\left(\psi_0^{-1}(\mathfrak{t}_\lambda) - \psi_0(\mathfrak{q})\psi_0^{-1}(\mathfrak{t}_{\lambda\mathfrak{q}})\right)L(\eta\psi_0^{-1}, 0) = 0,$$

we obtain the desired result for imprimitive ψ as well. $\square$

We will also need to compute the constant terms in the Fourier expansions of certain Eisenstein series at particular cusps different from ∞. More precisely, let

$$A = (A_\lambda) = \begin{pmatrix} 1 & x_\lambda \\ \alpha_\lambda & y_\lambda \end{pmatrix}_{\lambda \in \mathrm{Cl}^+(F)} \in \mathbf{SL}_2(F)^h, \tag{60}$$

where x_λ, y_λ, and α_λ are chosen such that

$$\alpha_\lambda \in \mathfrak{p}\mathfrak{d}\mathfrak{t}_\lambda, \qquad x_\lambda \in \mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}, \qquad \text{and} \qquad y_\lambda \in \mathfrak{n}.$$

Recall that for any Hilbert modular form $f = (f_\lambda)$, the slash operator is defined by

$$f|_A := (f_\lambda|_{A_\lambda}).$$

Proposition 2.3. *Fix $k > 2$. Suppose that $\mathfrak{p}$ is the only prime of F above p. Let η and ψ be narrow ray class characters of conductors $\mathfrak{a}$ and $\mathfrak{b}$, respectively, with $\mathfrak{a}\mathfrak{b} = \mathfrak{n}\mathfrak{p}$ and $\eta\psi = \chi\omega^{1-k}$. Then*

$$c_\lambda(0, E_k(\eta,\psi)|_A) = 0$$

unless $\mathfrak{a} = \mathfrak{n}$, $\mathfrak{b} = \mathfrak{p}$, and there exists a narrow ray class character ν of conductor 1 such that

$$\eta = \chi \cdot \nu^{-1}, \qquad \psi = \omega^{1-k} \cdot \nu.$$

In this case, we have

$$c_\lambda(0, E_k(\eta,\psi)|_A) = \mathrm{sgn}(\alpha_\lambda)\chi(\alpha_\lambda\mathfrak{p}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{d}^{-1}) \cdot \frac{\nu(\mathfrak{t}_\lambda/\mathfrak{n}^2)}{\langle \mathrm{N}\mathfrak{n}\rangle^{k-1}} \cdot \frac{2^{-n}}{\tau(\chi^{-1})} \cdot L_S(\chi^{-1}\omega^{1-k}\nu^2, 1-k).$$

Proof. Write $E_k(\eta,\psi) = (f_\lambda)$. For $k > 2$, the series (53) with $s = 0$ converges and yields f_λ. If for each $a \in \mathfrak{r}$ and $b \in \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{r}$ we define u and v by the matrix equation

$$\begin{pmatrix} a & b \end{pmatrix} A_\lambda = \begin{pmatrix} u & v \end{pmatrix}, \tag{61}$$

then

$$f_\lambda|_{A_\lambda}(z) = C \cdot \frac{\mathrm{N}(\mathfrak{t}_\lambda)^{-k/2}}{\mathrm{N}\mathfrak{b}} \sum_{\mathfrak{r},a,b} \mathrm{N}\mathfrak{r}^k \cdot \frac{\mathrm{sgn}(a)^q \eta(a\mathfrak{r}^{-1})}{(uz+v)^k} \sum_c \mathrm{sgn}(c)^r \psi(c\mathfrak{t}_\lambda^{-1}\mathfrak{r}) e_F(-bc), \tag{62}$$

where the indices of the sums are as in (53). To compute the unnormalized constant term of (62), we send each $z_i \to \infty i$; the only contribution arises when $u = 0$. This occurs when $a = -b\alpha_\lambda$ and $v = b$. By the choice of α_λ, the conditions $b\alpha_\lambda \in \mathfrak{r}$ and $b \in \mathfrak{d}^{-1}\mathfrak{b}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{r}$ imply that $b \in \mathfrak{d}^{-1}(\mathfrak{p},\mathfrak{b})^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{r}$. If $\mathfrak{p} \nmid \mathfrak{b}$ then $\mathfrak{p} \mid \mathfrak{a}$ and $b\alpha_\lambda\mathfrak{r}^{-1} \subset \mathfrak{p}$; thus $\eta(a\mathfrak{r}^{-1}) = 0$, and the constant term of $f_\lambda|_{A_\lambda}$ is 0. We therefore suppose $\mathfrak{p} \mid \mathfrak{b}$, and obtain for the constant term

$$C \cdot \frac{\mathrm{N}(\mathfrak{t}_\lambda)^{-k/2}\,\mathrm{sgn}(\alpha_\lambda)^q \eta(\alpha_\lambda\mathfrak{p}^{-1}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1})}{\mathrm{N}\mathfrak{b}} \sum_{\mathfrak{r}\in \mathrm{Cl}(F)} \mathrm{N}\mathfrak{r}^k \sum_{\substack{b \in \mathfrak{d}^{-1}\mathfrak{p}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{r}/U \\ b \neq 0}} \frac{\mathrm{sgn}(-b)^q \eta(-b\mathfrak{d}\mathfrak{p}\mathfrak{t}_\lambda\mathfrak{r}^{-1})}{\mathrm{N}b^k} \times$$
$$\sum_{c \in \mathfrak{t}_\lambda\mathfrak{r}^{-1}/\mathfrak{t}_\lambda\mathfrak{r}^{-1}\mathfrak{b}} \mathrm{sgn}(c)^r \psi(c\mathfrak{t}_\lambda^{-1}\mathfrak{r}) e_F(-bc). \tag{63}$$

If $\mathfrak{b} \neq \mathfrak{p}$, then a prime $\mathfrak{q} \neq \mathfrak{p}$ divides the conductor of ψ since $\mathfrak{ab} = \mathfrak{np}$ and $\eta\psi = \chi\omega^{1-k}$ is primitive away from $\mathfrak{p}$. For $c \in \mathfrak{t}_\lambda\mathfrak{r}^{-1}/\mathfrak{t}_\lambda\mathfrak{r}^{-1}\mathfrak{b}$ in a fixed residue class modulo $\mathfrak{t}_\lambda\mathfrak{r}^{-1}\mathfrak{p}$, the value $e_F(-bc)$ is constant; the existence of $\mathfrak{q}$ dividing the conductor of ψ implies that the sum of $\operatorname{sgn}(c)^r\psi(c\mathfrak{t}_\lambda^{-1}\mathfrak{r})$ over the c in each such class is 0.

Therefore, we are reduced to considering the situation $\mathfrak{b} = \mathfrak{p}$, $\mathfrak{a} = \mathfrak{n}$. Now $\eta = \chi\omega^{1-k}\psi^{-1}$ has conductor indivisible by $\mathfrak{p}$. Therefore $\psi = \omega^{1-k}\nu$ for some narrow ray class character ν of conductor 1, and $\eta = \chi\nu^{-1}$. When $k \not\equiv 1 \pmod{(p-1)}$, the character ψ is primitive, and we may apply (55). Then an argument following the proof of Proposition 2.1 using the change of variables $\mathfrak{f} = (b)\mathfrak{d}\mathfrak{b}\mathfrak{t}_\lambda\mathfrak{r}^{-1}$ and the functional equation for $L(\eta\psi^{-1}, k)$ shows that the value of (63) is equal to

$$\mathrm{N}(\mathfrak{t}_\lambda)^{k/2} \cdot \operatorname{sgn}(\alpha_\lambda)\chi(\alpha_\lambda\mathfrak{p}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{d}^{-1}) \cdot \frac{\nu(\mathfrak{t}_\lambda/\mathfrak{n}^2)}{\langle \mathrm{N}\mathfrak{n}\rangle^{k-1}} \cdot \frac{2^{-n}}{\tau(\chi^{-1})} \cdot L_S(\chi^{-1}\omega^{1-k}\nu^2, 1-k). \qquad (64)$$

When $k \equiv 1 \pmod{(p-1)}$, the sum over c in (63) is easy to evaluate directly, and via the same technique one again arrives at (64). $\square$

For weight $k = 1$, the analogue of Proposition 2.3 requires a separate treatment.

Proposition 2.4. *Suppose that $\mathfrak{p}$ is the only prime of F above p and that $\mathfrak{n} \neq 1$. Then*

$$c_\lambda(0, E_1(1,\chi)|_A) = \operatorname{sgn}(\alpha_\lambda)\chi(\alpha_\lambda\mathfrak{p}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{d}^{-1})\frac{2^{-n}}{\tau(\chi^{-1})}L_R(\chi^{-1}, 0).$$

Proof. Since χ is primitive of conductor $\mathfrak{n}$, the function $f_\lambda|_{A_\lambda}(z)$ is given by the value of the function

$$C \cdot \frac{\mathrm{N}(\mathfrak{t}_\lambda)^{-1/2}}{\mathrm{N}\mathfrak{n}}|\alpha_\lambda z + y_\lambda|^{2s} \cdot \sum_{\mathfrak{r},a,b} \mathrm{N}\mathfrak{r} \cdot \frac{\operatorname{sgn}(-b)\chi^{-1}(-b\mathfrak{d}\mathfrak{n}\mathfrak{t}_\lambda\mathfrak{r}^{-1})\tau(\chi)}{(uz+v)|uz+v|^{2s}}. \qquad (65)$$

at $s = 0$. Here the indices of the sum are as in (53) and (u, v) is defined in (61). The map $(a, b) \mapsto (u, v)$ induces a bijection between the sets

$$\left(\mathfrak{r} \times \mathfrak{d}^{-1}\mathfrak{n}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{r} - \{(0,0)\}\right) / \, U$$

and

$$\left(\mathfrak{r}\mathfrak{n}^{-1} \times \mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{r} - \{(0,0)\}\right) / \, U.$$

Furthermore, since $b = -x_\lambda u + v$ and $v\mathfrak{d}\mathfrak{n}\mathfrak{t}_\lambda\mathfrak{r}^{-1} \in \mathfrak{n}$, we have

$$\begin{aligned}\operatorname{sgn}(-b)\chi^{-1}(-b\mathfrak{d}\mathfrak{n}\mathfrak{t}_\lambda\mathfrak{r}^{-1}) &= \operatorname{sgn}(x_\lambda u)\chi^{-1}(x_\lambda u\mathfrak{d}\mathfrak{n}\mathfrak{t}_\lambda\mathfrak{r}^{-1}) \\ &= (-1)^n \operatorname{sgn}(\alpha_\lambda)\chi(\alpha_\lambda\mathfrak{p}^{-1}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}) \cdot \operatorname{sgn}(u)\chi^{-1}(u\mathfrak{n}\mathfrak{r}^{-1}),\end{aligned}$$

where the last equality uses $x_\lambda\alpha_\lambda \equiv -1 \pmod{\mathfrak{n}}$ and $\chi(\mathfrak{p}) = 1$.

Let $\mathfrak{r}' = \mathfrak{r}\mathfrak{n}^{-1}$ and $\mathfrak{t}'_\lambda = \mathfrak{t}_\lambda\mathfrak{n}^{-1}$ denote new representatives for $\mathrm{Cl}(F)$ and $\mathrm{Cl}^+(F)$, respectively. Thus the function in (65) may be written as the product of the constant

$$\mathrm{N}\mathfrak{n}^{-1/2} \cdot (-1)^n \operatorname{sgn}(\alpha_\lambda)\chi(\alpha_\lambda\mathfrak{p}^{-1}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}) \cdot \tau(\chi) \qquad (66)$$

and the function

$$C \cdot \mathrm{N}(\mathfrak{t}'_\lambda)^{-1/2} |\alpha_\lambda z + y_\lambda|^{2s} \cdot \sum_{\mathfrak{r}', u, v} \mathrm{N}\mathfrak{r}' \cdot \frac{\operatorname{sgn}(u)\chi^{-1}(u(\mathfrak{r}')^{-1})}{(uz+v)|uz+v|^{2s}}, \tag{67}$$

where (u, v) ranges over representatives for

$$\left(\mathfrak{r}' \times \mathfrak{d}^{-1}(\mathfrak{t}'_\lambda)^{-1}\mathfrak{r}' - \{(0,0)\}\right) / \, U.$$

The expression in (67) at $s = 0$ is precisely the $\lambda\mathfrak{n}^{-1}$ component of $E_1(\chi^{-1}, 1)$ given the representatives $\mathfrak{t}'_\lambda$ for $\mathrm{Cl}^+(F)$. Therefore, its unnormalized constant term is equal to

$$\frac{\mathrm{N}(\mathfrak{t}'_\lambda)^{1/2}}{2^n} L(\chi^{-1}, 0). \tag{68}$$

Using the fact that

$$\tau(\chi)\tau(\chi^{-1}) = (-1)^n \, \mathrm{N}\mathfrak{n}$$

and that $\mathrm{N}(\mathfrak{t}'_\lambda) = \mathrm{N}(\mathfrak{t}_\lambda) / \, \mathrm{N}\mathfrak{n}$, the expressions (66) and (68) combine to show that the normalized constant term of $f_\lambda|_{A_\lambda}$ is

$$\operatorname{sgn}(\alpha_\lambda)\chi(\alpha_\lambda \mathfrak{p}^{-1}\mathfrak{d}^{-1}\mathfrak{t}_\lambda^{-1}) \cdot \frac{2^{-n}}{\tau(\chi^{-1})} L_R(\chi^{-1}, 0)$$

as desired. □

When $L(\psi, 1-k) \neq 0$, we define the normalized Eisenstein series

$$G_k(1, \psi) := \frac{2^n}{L(\psi, 1-k)} E_k(1, \psi), \tag{69}$$

each of whose constant coefficients $c_\lambda(G_k(1, \psi), 0)$ is equal to 1. The Eisenstein series $G_{k-1}(1, \omega^{1-k})$ plays an important role in our constructions. The following proposition computes its constant term at the cusp $A\infty$, where A is given in (60).

Proposition 2.5. *Suppose that $\mathfrak{p}$ is the only prime of F above p. For $k > 2$ we have*

$$c_\lambda(0, G_{k-1}(1, \omega^{1-k})|_A) = 1.$$

Proof. We must show that

$$c_\lambda(0, E_{k-1}(1, \omega^{1-k})|_A) = \frac{1}{2^n} L(\omega^{1-k}, 2-k).$$

The argument used in the proof of Proposition 2.3 with $(\chi, \mathfrak{n}, \chi\omega^{1-k})$ replaced by $(1, 1, \omega^{1-k})$ gives the desired equality. The details are left to the reader. □

2.3 A product of Eisenstein series

Recall that χ is a primitive character of conductor $\mathfrak{n}$ and that $\mathfrak{p} \mid p$ satisfies $\chi(\mathfrak{p}) = 1$. Set

$$S = \{\lambda \mid \mathfrak{n}p\infty\}, \quad R = S - \{\mathfrak{p}\}.$$

Adjoining a finite prime $\mathfrak{q} \nmid \mathfrak{n}p$ to S multiplies $L'_{S,p}(\chi\omega, 0)$ and $L_R(\chi, 0)$ by $1 - \chi(\mathfrak{q})$, and leaves $\mathscr{L}(\chi)$ unchanged when $\chi(\mathfrak{q}) \neq 1$, so it suffices to prove Theorem 2 with this minimal choice of S. Let p' denote the ideal of F given by

$$p' = \prod_{\mathfrak{q}|p,\ \mathfrak{q}\neq\mathfrak{p}} \mathfrak{q},$$

and define

$$\mathfrak{n}_R = \operatorname{lcm}(\mathfrak{n}, p'), \quad \mathfrak{n}_S = \operatorname{lcm}(\mathfrak{n}, p'\mathfrak{p}).$$

We will denote by χ_R the character χ viewed as having modulus $\mathfrak{n}_R$ (in particular, we have $\chi_R(\mathfrak{q}) = 0$ for all $\mathfrak{q} \mid p'$). We always view the character $\chi\omega^{1-k}$ as having modulus $\mathfrak{n}_S$, even when it has smaller conductor. In particular, $\chi\omega^0 \neq \chi$, since $\chi\omega^0(\mathfrak{p}) = 0$ while $\chi(\mathfrak{p}) = 1$. Occasionally, we will write χ_S instead of $\chi\omega^0$.

In this section, we consider the modular form

$$P_k := E_1(1, \chi_R) \cdot G_{k-1}(1, \omega^{1-k}) \in M_k(\mathfrak{n}_S, \chi\omega^{1-k}). \tag{70}$$

It is a fact that every modular form in $M_k(\mathfrak{n}_S, \chi\omega^{1-k})$ can be written uniquely as a linear combination of a cusp form and the Eisenstein series $E_k(\eta, \psi)$, where (η, ψ) run over the set J of pairs of (possibly imprimitive) characters of modulus m_η and m_ψ respectively, satisfying

$$m_\eta m_\psi = \mathfrak{n}_S, \qquad \eta\psi = \chi\omega^{1-k}.$$

(For $k = 2$, this is explained in [15, Prop. 1.5], and the case of general $k \geq 2$ follows from the same argument.) For each $(\eta, \psi) \in J$, let $a_k(\eta, \psi) \in E$ be the unique constant such that

$$P_k = \begin{pmatrix} \text{A cusp} \\ \text{form} \end{pmatrix} + \sum_{(\eta,\psi)\in J} a_k(\eta, \psi) E_k(\eta, \psi). \tag{71}$$

We will be particularly interested in the coefficients $a_k(1, \chi\omega^{k-1})$ and $a_k(\chi, \omega^{k-1})$ that appear in this linear combination. It turns out that these coefficients can be expressed as ratios of special values of the classical L-function $L_R(\chi, s)$ and of the p-adic L-function $L_{S,p}(\chi\omega, s)$.

Proposition 2.6. *For all integers $k \geq 2$, we have*

$$a_k(1, \chi\omega^{1-k}) = \frac{L_R(\chi, 0)}{L_{S,p}(\chi\omega, 1-k)} = -\mathscr{L}_{\mathrm{an}}(\chi, k)^{-1}.$$

Proof. The result follows by comparing the constant terms on both sides of (71). More precisely, since the constant terms of $G_{k-1}(1, \omega^{1-k})$ are equal to 1, it follows that

$$c_\lambda(0, P_k) = c_\lambda(0, E_1(1, \chi_R)) = 2^{-n} \cdot \begin{cases} L_R(\chi, 0) & \text{if } \mathfrak{n}_R \neq 1 \\ L_R(\chi, 0) + \chi^{-1}(\mathfrak{t}_\lambda) L_R(\chi^{-1}, 0) & \text{if } \mathfrak{n}_R = 1. \end{cases} \tag{72}$$

On the other hand, the constant terms of $E_k(\eta, \psi)$ for $(\eta, \psi) \in J$ are equal to 0 except for the h characters η with $m_\eta = 1$. For such η, the h-tuple of constant terms $c_\lambda(0, E_k(\eta, \psi))$ is equal to

$$2^{-n} L_S(\eta^{-2}\chi\omega^{1-k}, 1-k) \times (\eta^{-1}(\mathfrak{t}_{\lambda_1}), \dots, \eta^{-1}(\mathfrak{t}_{\lambda_h})).$$

The linear independence of the characters of $\mathrm{Cl}^+(F)$ implies that these h-tuples are linearly independent. It follows that $a_k(\eta, \psi) = 0$ for all unramified $\eta \notin \{1, \chi\}$, and that

$$a_k(1, \chi\omega^{1-k}) = \frac{L_R(\chi, 0)}{L_{S,p}(\chi\omega, 1-k)},$$

as desired. □

Consider now the special case where $\mathfrak{p}$ is the unique prime of F lying above p, so in particular $\chi_R = \chi$ is a primitive character of conductor $\mathfrak{n}_R = \mathfrak{n}$. We will be interested in the coefficient $a_k(\chi, \omega^{1-k})$ of $E_k(\chi, \omega^{1-k})$ in the linear combination (71). If $\mathfrak{n}_R = \mathfrak{n} = 1$, observe that the $\eta = \chi$ coefficient in the proof of Proposition 2.6 yields

$$a_k(\chi, \omega^{1-k}) = \frac{L_R(\chi^{-1}, 0)}{L_{S,p}(\chi^{-1}\omega, 1-k)}. \tag{73}$$

The following proposition shows that a similar formula holds more generally whenever $\mathfrak{p}$ is the unique prime of F lying above p.

Proposition 2.7. *Suppose that $\mathfrak{p}$ is the unique prime above p in F. For an integer $k > 2$, we have*

$$a_k(\chi, \omega^{1-k}) = \frac{L_R(\chi^{-1}, 0)}{L_{S,p}(\chi^{-1}\omega, 1-k)} \cdot \langle \mathrm{N}\mathfrak{n} \rangle^{k-1} = -\mathscr{L}_{\mathrm{an}}(\chi^{-1}, k)^{-1} \cdot \langle \mathrm{N}\mathfrak{n} \rangle^{k-1}.$$

Proof of Proposition 2.7. Since we have proven the result in the case $\mathfrak{n} = 1$ in equation (73) already, we assume $\mathfrak{n} \neq 1$. The computation of $a_k(\chi, \omega^{1-k})$ will proceed by slashing equation (71) with the collection $A = (A_\lambda)$ of matrices given in (60), and then comparing constant terms.

By Proposition 2.3, we know that $c_\lambda(E_k(\eta, \psi)|_A) = 0$ unless $\mathfrak{a} = \mathfrak{n}$, $\mathfrak{b} = \mathfrak{p}$, and the characters η and ψ are of the form

$$\eta = \chi \cdot \nu^{-1}, \qquad \psi = \omega^{1-k} \cdot \nu, \tag{74}$$

where ν is a narrow ray class character of conductor 1. In this remaining case, Proposition 2.3 asserts that

$$c_\lambda(0, E_k(\eta, \psi)|_A) = \mathrm{sgn}(\alpha_\lambda)\chi(\alpha_\lambda \mathfrak{p}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{d}^{-1}) \cdot \frac{\nu(\mathfrak{t}_\lambda/\mathfrak{n}^2)}{\langle \mathrm{N}\mathfrak{n} \rangle^{k-1}} \cdot \frac{2^{-n}}{\tau(\chi^{-1})} \cdot L_S(\chi^{-1}\omega^{1-k}\nu^2, 1-k).$$

As in the proof of Proposition 2.6, we see that for the h unramified characters ν corresponding to the pairs (η, ψ) of the form (74), the h-tuples $c_\lambda(E_k(\eta,\psi)|_A))$ are constant multiples of the vectors $(\nu(\mathfrak{t}_{\lambda_1}), \ldots, \nu(\mathfrak{t}_{\lambda_h}))$. By the linear independence of these h-tuples, the coefficients $a_k(\eta, \psi)$ can once again be read off from the constant terms of $P_k|_A$. But it follows from Propositions 2.3 through 2.5 that

$$c_\lambda(P_k|_A) = \mathrm{sgn}(\alpha_\lambda)\chi(\alpha_\lambda \mathfrak{p}^{-1}\mathfrak{t}_\lambda^{-1}\mathfrak{d}^{-1}) \cdot \frac{2^{-n}}{\tau(\chi^{-1})} L_R(\chi^{-1}, 0).$$

Therefore we conclude that

$$a_k(\chi, \omega^{1-k}) = \frac{L_R(\chi^{-1}, 0)}{L_{S,p}(\chi^{-1}\omega, 1-k)} \cdot \langle \mathrm{N}\mathfrak{n} \rangle^{k-1}$$

and that $a_k(\eta, \psi) = 0$ for (η, ψ) as in (74) when $\nu \neq 1$. □

2.4 The ordinary projection

Recall that E is a finite extension of $\mathbf{Q}_p$ containing the values of the character χ, and that we have fixed an embedding $\overline{\mathbf{Q}}_p \subset \mathbf{C}$. Let $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; \mathcal{O}_E)$ denote the $\mathcal{O}_E$-submodule of $M_k(\mathfrak{n}_S, \chi\omega^{1-k})$ consisting of modular forms with all normalized Fourier coefficients lying in $\mathcal{O}_E$. The ordinary projection operator

$$e = \lim_{r \to \infty} (\prod_{\mathfrak{q}|p} U_\mathfrak{q})^{r!}$$

gives rise to an idempotent in the endomorphism ring of $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; \mathcal{O}_E)$. We extend it to $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; E)$ by E-linearity, using the fact that

$$M_k(\mathfrak{n}_S, \chi\omega^{1-k}; E) = M_k(\mathfrak{n}_S, \chi\omega^{1-k}; \mathcal{O}_E) \otimes_{\mathcal{O}_E} E.$$

The projection e preserves the cusp forms as well as the lines spanned by the various Eisenstein series $E_k(\eta, \psi)$ with $(\eta, \psi) \in J$. More precisely, if $\mathfrak{q} \mid p$, then $\mathfrak{q} \mid \mathfrak{n}_S = m_\eta m_\psi$, so we have:

$$\begin{aligned} U_\mathfrak{q} E_k(\eta,\psi) &= (\eta(\mathfrak{q}) + \psi(\mathfrak{q})(\mathrm{N}\mathfrak{q})^{k-1}) E_k(\eta,\psi) \\ &= \left\{ \begin{array}{ll} \eta(\mathfrak{q}) & \text{if } \mathfrak{q} \nmid m_\eta \\ \psi(\mathfrak{q})(\mathrm{N}\mathfrak{q})^{k-1} & \text{if } \mathfrak{q} \mid m_\eta \end{array} \right\} \times E_k(\eta,\psi). \end{aligned}$$

It follows that for $k \geq 2$:

$$eE_k(\eta,\psi) = \begin{cases} E_k(\eta,\psi) & \text{if } \gcd(p, m_\eta) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{75}$$

Let

$$P_k^o := eP_k \tag{76}$$

denote the projection of P_k to the ordinary subspace.

Proposition 2.8. *The modular form P_k^o can be written*

$$P_k^o = \begin{pmatrix} \textit{An ordinary} \\ \textit{cusp form} \end{pmatrix} + \sum_{(\eta,\psi)\in J^o} a_k(\eta,\psi)E_k(\eta,\psi), \tag{77}$$

where (η,ψ) ranges over the set J^o of pairs $(\eta,\psi)\in J$ satisfying

$$m_\eta m_\psi = \mathfrak{n}_S, \quad \eta\psi = \chi\omega^{1-k}, \quad (p, m_\eta) = 1.$$

Note that for $(\eta,\psi) = (1,\chi\omega^{1-k})$ and (χ,ω^{1-k}), the coefficients $a_k(\eta,\psi)$ are the ones given in Propositions 2.6 and 2.7, respectively (the latter only in the case where $\mathfrak{p}$ is the only prime of F above p).

2.5 Construction of a cusp form

We begin by defining a modular form H_k as in the introduction, by taking a suitable linear combination of the Eisensein series $E_k(1,\chi\omega^{1-k})$ and $E_k(\chi,\omega^{1-k})$ and the form P_k^o. It is useful to distinguish two cases.

Case 1. *The set R contains a prime above p, i.e. F has a prime above p other than $\mathfrak{p}$.* This case is the simplest of the two, but did not arise in the introduction since $\mathfrak{p} = (p)$ when $F = \mathbf{Q}$. We set

$$H_k := u_k E_k(1,\chi\omega^{1-k}) + w_k P_k^o, \tag{78}$$

where

$$u_k := \frac{1}{1+\mathscr{L}_{\mathrm{an}}(\chi,k)}, \qquad w_k := \frac{\mathscr{L}_{\mathrm{an}}(\chi,k)}{1+\mathscr{L}_{\mathrm{an}}(\chi,k)}. \tag{79}$$

Note that the vector (u_k, w_k) is proportional to the vector $(a_k(1,\chi\omega^{1-k}),-1)$, but has been normalized to that $u_k + w_k = 1$.

Case 2. *The set R contains no primes above p, i.e. $\mathfrak{p}$ is the unique prime of F above p.* In this case the definition of H_k involves the Eisenstein series $E(\chi,\omega^{k-1})$ as well:

$$H_k := u_k E_k(1,\chi\omega^{1-k}) + v_k E_k(\chi,\omega^{1-k}) + w_k P_k^o, \tag{80}$$

where

$$u_k := \frac{\mathscr{L}_{\mathrm{an}}(\chi,k)^{-1}}{c_k}, \quad v_k := \frac{\mathscr{L}_{\mathrm{an}}(\chi^{-1},k)^{-1}\langle \mathrm{N}\mathfrak{n}\rangle^{k-1}}{c_k}, \quad w_k := \frac{1}{c_k}, \tag{81}$$

with

$$c_k := \mathscr{L}_{\mathrm{an}}(\chi,k)^{-1} + \mathscr{L}_{\mathrm{an}}(\chi^{-1},k)^{-1}\langle \mathrm{N}\mathfrak{n}\rangle^{k-1} + 1.$$

In this case, the vector (u_k, v_k, w_k) is proportional to $(a_k(1,\chi\omega^{1-k}), a_k(\chi,\omega^{1-k}), -1)$ and is normalized so that $u_k + v_k + w_k = 1$.

It follows that the modular form $H_k \in M_k(\mathfrak{n}_S, \chi\omega^{1-k})$ is a linear combination of a cusp form and the Eisenstein series $E_k(\eta,\psi)$, where (η,ψ) ranges over the elements J^o with $\eta\neq 1$ in case 1, and $\eta \neq 1,\chi$ in case 2.

Lemma 2.9. *We have the following:*

1. *For each $(\eta, \psi) \in J^o$ with $\eta \notin \{1, \chi\}$, there is a Hecke operator $T_{(\eta,\psi)}$ satisfying*
$$T_{(\eta,\psi)} E_k(\eta, \psi) = 0, \qquad T_{(\eta,\psi)} E_1(1, \chi_S) \neq 0.$$

2. *In case 1, there is a Hecke operator $T_{(\chi,\omega^{1-k})}$ satisfying*
$$T_{(\chi,\omega^{1-k})} E_k(\chi, \omega^{1-k}) = 0, \qquad T_{(\chi,\omega^{1-k})} E_1(1, \chi_S) \neq 0.$$

Proof. Let us write $\eta \sim \eta'$ if the two characters η and η' agree on all but finitely many primes, i.e., if they come from the same primitive character.

Let (η, ψ) be any element of J^o. If $\eta \sim 1$, then $\psi \sim \chi\omega^{1-k}$ and the condition $(\eta, \psi) \in J^o$ implies that ψ has conductor $\mathfrak{n}_S$. Therefore η has conductor 1 and $\eta = 1$. If $\eta \sim \chi$, then η is necessarily of modulus $\mathfrak{n}$ when $(\mathfrak{n}, p) = 1$, and hence $\eta = \chi$. Hence, in proving part 1 we may assume that $\eta \not\sim 1, \chi$. Let $H_{(\eta,\psi)}$ be the abelian extension of F that is cut out by η and ψ by class field theory, and view η and ψ as characters of $\mathrm{Gal}(H_{(\eta,\psi)}/F)$. Let σ be any element of $\mathrm{Gal}(H_{(\eta,\psi)}/F)$ that does not belong to $\ker(\eta) \cup \ker(\eta\chi^{-1})$. Such an element exists because a group cannot be the union of two of its proper subgroups. By the Chebotarev density theorem, there is a prime $\lambda \nmid \mathfrak{n}_S$ of F whose Frobenius element is equal to σ. For such a prime we have
$$\{\eta(\lambda), \psi\omega^{k-1}(\lambda)\} \neq \{1, \chi(\lambda)\}, \qquad \eta(\lambda) \times \psi\omega^{k-1}(\lambda) = 1 \times \chi(\lambda).$$
It follows that
$$\eta(\lambda) + \psi\omega^{k-1}(\lambda) \neq 1 + \chi(\lambda).$$
Now, set
$$\begin{aligned} T_{(\eta,\psi)} &:= T_\lambda - \eta(\lambda) - \psi(\lambda)(\mathrm{N}\lambda)^{k-1} \\ &= T_\lambda - \eta(\lambda) - \psi\omega^{k-1}(\lambda)\langle \mathrm{N}\lambda \rangle^{k-1}. \end{aligned} \tag{82}$$
It is easy to see that the operator $T_{(\eta,\psi)}$ has the desired properties, in light of the fact that
$$\begin{aligned} T_\lambda E_1(1, \chi_S) &= (1 + \chi(\lambda)) E_1(1, \chi_S), \\ T_\lambda E_k(\eta, \psi) &= (\eta(\lambda) + \psi(\lambda)(\mathrm{N}\lambda)^{k-1}) E_k(\eta, \psi). \end{aligned}$$
This proves part 1. Part 2 is proved by choosing a prime $\mathfrak{q}$ of F above p different from $\mathfrak{p}$, and setting
$$T_{(\chi,\omega^{1-k})} := U_{\mathfrak{q}} - \chi(\mathfrak{q}). \tag{83}$$
Since $U_{\mathfrak{q}}$ acts with eigenvalue $\chi(\mathfrak{q})$ on $E_k(\chi, \omega^{1-k})$ and with eigenvalue 1 on $E_1(1, \chi_S)$, the result follows (recall that $\chi(\mathfrak{q}) \neq 1$ by Lemma 1.1). ∎

Corollary 2.10. *The modular form*
$$F_k := \left(\prod_{(\eta,\psi)} T_{(\eta,\psi)} \right) H_k,$$
where the product is taken over all $(\eta, \psi) \in J^o$ with $\eta \neq 1$ in case 1 and $\eta \neq 1, \chi$ in case 2, belongs to $S_k(\mathfrak{n}_S, \chi\omega^{1-k})$.

3 Λ-adic forms

3.1 Definitions

We now recall the basic definitions of Λ-adic forms. Let χ be any totally odd E-valued narrow ray class character of F modulo $\mathfrak{n}$. Recall from the introduction the Iwasawa algebra $\Lambda \simeq \mathcal{O}_E[\![T]\!]$, which is topologically generated over $\mathcal{O}_E$ by functions of the form $k \mapsto a^k$ with $a \in 1 + p\mathbf{Z}_p$. For each $k \in \mathbf{Z}_p$, write

$$\nu_k : \Lambda \to \mathcal{O}_E$$

for the evaluation map at k. We will write $\Lambda_{(k)} \subset \mathcal{F}_\Lambda$ for the localization of Λ at $\ker \nu_k$, and sometimes view ν_k as a homomorphism from $\Lambda_{(k)}$ to E.

In Section 2, we recalled the definition of the space $M_k(\mathfrak{n}_S, \chi\omega^{1-k})$ of classical Hilbert modular forms of weight k, level $\mathfrak{n}_S$, and character $\chi\omega^{1-k}$. To each such form f is associated its set of normalized Fourier coefficients $c(\mathfrak{m}, f)$, indexed by the nonzero integral ideals of F, and normalized constant terms $c_\lambda(0, f)$, indexed by the narrow class group $\mathrm{Cl}^+(F)$.

Definition 3.1. A Λ-adic form $\mathscr{F}$ of level $\mathfrak{n}$ and character χ is a collection of elements of Λ:

$$\begin{cases} c(\mathfrak{m}, \mathscr{F}) \text{ for all nonzero integral ideals } \mathfrak{m} \text{ of } F \\ c_\lambda(0, \mathscr{F}) \text{ for } \lambda \in \mathrm{Cl}^+(F), \end{cases}$$

such that for all but finitely many $k \geq 2$, there is an element of $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; E)$ with normalized Fourier coefficients $\nu_k(c(\mathfrak{m}, \mathscr{F}))$ for nonzero integral ideals $\mathfrak{m}$, and normalized constant terms $\nu_k(c_\lambda(0, \mathscr{F}))$ for $\lambda \in \mathrm{Cl}^+(F)$.

The space of Λ-adic forms of level $\mathfrak{n}$ and character χ is denoted $\mathcal{M}(\mathfrak{n}, \chi)$. A Λ-adic form $\mathscr{F}$ is said to be a *cusp form* if $\nu_k(\mathscr{F})$ belongs to $S_k(\mathfrak{n}_S, \chi\omega^{1-k})$ for almost all $k \geq 2$, and the space of such cusp forms is denoted $\mathcal{S}(\mathfrak{n}, \chi)$. The action of the Hecke operators T_λ for $\lambda \nmid \mathfrak{n}p$ and $U_\mathfrak{q}$ for $\mathfrak{q} \mid \mathfrak{n}p$ on $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; E)$ lifts to a Λ-linear action of these operators on $\mathcal{M}(\mathfrak{n}, \chi)$ that preserves the subspace $\mathcal{S}(\mathfrak{n}, \chi)$ (see §1.2 of [16]).

Hida's ordinary projection can still be defined by

$$e = \lim_{r \to \infty} (\prod_{\mathfrak{q} \mid p} U_\mathfrak{q})^{r!}.$$

The image of e is the submodule of ordinary forms:

$$\mathcal{M}^o(\mathfrak{n}, \chi) := e\mathcal{M}(\mathfrak{n}, \chi), \qquad \mathcal{S}^o(\mathfrak{n}, \chi) := e\mathcal{S}(\mathfrak{n}, \chi).$$

The spaces $\mathcal{M}^o(\mathfrak{n}, \chi)$ and $\mathcal{S}^o(\mathfrak{n}, \chi)$ are finitely generated torsion-free Λ-modules. Let $\tilde{\mathbf{T}}$ and $\mathbf{T}$ denote the Λ-algebras of Hecke operators acting on $\mathcal{M}^o(\mathfrak{n}, \chi)$ and $\mathcal{S}^o(\mathfrak{n}, \chi)$ respectively.

By extension, we also call any element of $\mathcal{M}^o(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$ or $\mathcal{S}^o(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$ a Λ-adic modular form or Λ-adic cusp form, respectively. Note that if $\mathscr{F}$ is such an element, then its weight k specialization $\nu_k(\mathscr{F})$ is defined for almost all $k \in \mathbf{Z}_p$.

3.2 Λ-adic Eisenstein series

The most basic examples of Λ-adic forms are the Eisenstein series. Let η and ψ be a pair of narrow ray class characters modulo m_η and m_ψ respectively, such that $\eta\psi$ is totally odd.

Proposition 3.2. *There exists a Λ-adic modular form*

$$\mathscr{E}(\eta, \psi) \in \mathcal{M}(m_\eta m_\psi, \eta\psi) \otimes \mathcal{F}_\Lambda$$

such that

$$\nu_k(\mathscr{E}(\eta, \psi)) = E_k(\eta, \psi\omega^{1-k}).$$

Proof. The Eisenstein series $E_k(\eta, \psi\omega^{1-k})$ satisfy

$$c(\mathfrak{m}, E_k(\eta, \psi\omega^{1-k})) = \sum_{\substack{\mathfrak{r}|\mathfrak{m} \\ (\mathfrak{r},p)=1}} \eta\left(\frac{\mathfrak{m}}{\mathfrak{r}}\right)\psi(\mathfrak{r})\langle \mathrm{N}\mathfrak{r}\rangle^{k-1} \tag{84}$$

and

$$c_\lambda(0, E_k(\eta, \psi\omega^{1-k})) = \begin{cases} 2^{-n}\eta^{-1}(\mathfrak{t}_\lambda)L_p(\eta^{-1}\psi\omega, 1-k) & \text{if } m_\eta = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{85}$$

One sees by inspection that the expressions on the right of (84), viewed as functions of k, belong to the Iwasawa algebra Λ . That the function $k \mapsto L_p(\eta^{-1}\psi\omega, 1-k)$ belongs to $\mathcal{F}_\Lambda$ is equivalent to the known assertions about the existence and basic properties of the p-adic L-functions $L_p(\eta^{-1}\psi\omega, s)$ (see, for example, [17, (1.3)]). ☐

We remark in passing that the idea of realizing the special values $L_p(\eta^{-1}\psi\omega, 1-k)$ as the constant terms of Eisenstein series whose other Fourier coefficients lie in Λ is a key component of the construction of p-adic L-functions for totally real fields due to Serre and Deligne-Ribet.

Recall the normalized Eisenstein series $G_{k-1}(1, \omega^{1-k})$ with constant term 1. Its p-adic interpolation is most conveniently described in terms of the space $\mathcal{M}'$ of Λ-adic forms "with weights shifted by 1". An element of $\mathcal{M}'$ is a collection of elements $c(\mathfrak{m}, \mathscr{F})$ and $c_\lambda(0, \mathscr{F})$ of Λ with the property that the specializations $\{\nu_k(c(\mathfrak{m}, \mathscr{F})), \nu_k(c_\lambda(0, \mathscr{F}))\}$ are the normalized Fourier coefficients of an element in $M_{k-1}(p, \omega^{1-k}; E)$. There exists an element $\mathscr{G} \in \mathcal{M}' \otimes \mathcal{F}_\Lambda$ satisfying

$$\nu_k(\mathscr{G}) = G_{k-1}(1, \omega^{1-k}).$$

It is defined by the data:

$$c(\mathfrak{m}, \mathscr{G}) = 2^n \zeta_p(F, 2-k)^{-1} \sum_{\substack{\mathfrak{r}|\mathfrak{m} \\ (\mathfrak{r},p)=1}} (\mathrm{N}\mathfrak{r})^{-1}\langle \mathrm{N}\mathfrak{r}\rangle^{k-1}, \quad c_\lambda(0, \mathscr{G}) = 1, \tag{86}$$

where $\zeta_p(F, s) = L_p(1, s)$ is the p-adic zeta-function attached to F. We see by inspection that all of the coefficients of $\mathscr{G}$ belong to $\mathcal{F}_\Lambda$, and that $\nu_k(\mathscr{G}) = G_{k-1}$. The specialization of $\mathscr{G}$ at $k = 1$ plays a crucial role in our argument.

Proposition 3.3. *Assume Leopoldt's conjecture for F. The form $\mathscr{G}$ belongs to $\mathcal{M}' \otimes \Lambda_{(1)}$, and*

$$\nu_1(\mathscr{G}) = 1.$$

Proof. By a result of Colmez (cf. the main theorem of [2]), the non-vanishing of the p-adic regulator of F is known to imply that the p-adic zeta-function $\zeta_p(F, s)$ has a simple pole at $s = 1$. Therefore $\zeta_p(F, 2-k)^{-1}$ is regular at $k = 1$ and vanishes at that point. The result follows. □

3.3 A Λ-adic cusp form

In order to invoke Proposition 3.3, we assume Leopoldt's conjecture for F in the rest of Section 3. In case 2, we also assume that assumption (11) holds.

Recall the classical modular forms P_k, P_k^o and H_k in $M_k(\mathfrak{n}_S, \chi\omega^{1-k})$ that were defined in Sections 2.3 and 2.4 (see equations (70), (76), (78), and (80)), and the cusp form $F_k \in S_k(\mathfrak{n}_S, \chi\omega^{1-k})$ that was constructed in Corollary 2.10 in Section 2.5.

Proposition 3.4. *There exist Λ-adic forms $\mathscr{P} \in \mathcal{M}(\mathfrak{n}, \chi)\otimes\Lambda_{(1)}$, $\mathscr{P}^o$, $\mathscr{H} \in \mathcal{M}^o(\mathfrak{n}, \chi)\otimes\Lambda_{(1)}$, and $\mathscr{F} \in \mathcal{S}^o(\mathfrak{n}, \chi) \otimes \Lambda_{(1)}$ satisfying, for almost all $k \geq 2$:*

$$\nu_k(\mathscr{P}) = P_k, \quad \nu_k(\mathscr{P}^o) = P_k^o, \quad \nu_k(\mathscr{H}) = H_k, \quad \nu_k(\mathscr{F}) = F_k. \tag{87}$$

Proof. The forms $\mathscr{P}$ and $\mathscr{P}^o$ are simply defined by setting

$$\mathscr{P} = E_1(1, \chi_R)\mathscr{G}, \qquad \mathscr{P}^o = e\mathscr{P}.$$

To define the modular form $\mathscr{H}$, recall first the coefficients u_k, v_k and w_k that were defined in equations (79) and (81). In case 1, we set $v_k = 0$ so that in all cases we have $u_k+v_k+w_k = 1$. We observe that the coefficients u_k, v_k and w_k can be interpolated by elements $u, v, w \in \mathcal{F}_\Lambda$ satisfying $u(k) = u_k$, $v(k) = v_k$, and $w(k) = w_k$ for almost all $k \geq 2$, since the functions $\mathscr{L}_{\mathrm{an}}(\chi, k)^{-1}$ and $\mathscr{L}_{\mathrm{an}}(\chi^{-1}, k)^{-1}$ belong to $\mathcal{F}_\Lambda$. A direct calculation then shows that u, v and w belong to $\Lambda_{(1)}$. More precisely,

$$\begin{array}{llll} u(1) = 1; & v(1) = 0; & w(1) = 0 & \text{in case 1;} \\ u(1) = \frac{\mathscr{L}_{\mathrm{an}}^{(t)}(\chi^{-1},1)}{\mathscr{L}_{\mathrm{an}}^{(t)}(\chi,1)+\mathscr{L}_{\mathrm{an}}^{(t)}(\chi^{-1},1)}; & v(1) = \frac{\mathscr{L}_{\mathrm{an}}^{(t)}(\chi,1)}{\mathscr{L}_{\mathrm{an}}^{(t)}(\chi,1)+\mathscr{L}_{\mathrm{an}}^{(t)}(\chi^{-1},1)}; & w(1) = 0 & \text{in case 2,} \end{array} \tag{88}$$

where t denotes the common order of vanishing of $\mathscr{L}_{\mathrm{an}}(\chi, k) + \mathscr{L}_{\mathrm{an}}(\chi^{-1}, k)$ and $\mathscr{L}_{\mathrm{an}}(\chi^{-1}, k)$ at $k = 1$. It is at this stage that we use the hypothesis (11) appearing in Theorem 2. In both case 1 and case 2, it can be seen that the coefficients u, v and w belong to $\Lambda_{(1)}$ and that u is even invertible in this ring. In particular, the Λ-adic form

$$\mathscr{H} := u\mathscr{E}(1, \chi) + v\mathscr{E}(\chi, 1) + w\mathscr{P}^o$$

belongs to $\mathcal{M}^o(\mathfrak{n}, \chi) \otimes \Lambda_{(1)}$ and satisfies $\nu_k(\mathscr{H}) = H_k$.

Finally, we note that the Hecke operators $T_{(\eta,\psi)}$ of Lemma 2.9 that are defined in equations (82) and (83) can be viewed as elements of the ordinary Λ-adic Hecke algebra $\tilde{\mathbf{T}}$. We may therefore set

$$\mathscr{F} = \prod_{(\eta,\psi)} T_{(\eta,\psi)}\mathscr{H}$$

to obtain the desired Λ-adic cusp form satisfying $\nu_k(\mathscr{F}) = F_k$. Proposition 3.4 follows. $\square$

From now on, we will write u_1, v_1, and w_1 for $u(1), v(1)$, and $w(1)$, respectively. We now analyze the weight 1 specialization of the Λ-adic forms of Proposition 3.4.

Lemma 3.5. *We have*

$$\nu_1(\mathscr{P}) = \nu_1(\mathscr{P}^o) = E_1(1,\chi_R), \quad \nu_1(\mathscr{H}) = E_1(1,\chi_S),$$

and

$$\nu_1(\mathscr{F}) = t \cdot E_1(1,\chi_S), \qquad \textit{for some } t \in E^\times.$$

Proof. Proposition 3.3 directly implies that $\nu_1(\mathscr{P}) = E_1(1,\chi_R)$. (It is here that Leopoldt's conjecture is required in our argument.) To study $\nu_1(\mathscr{P}^o)$, we note that the operator $U_{\mathfrak{p}}$ preserves the two-dimensional subspace W_χ of $M_1(\mathfrak{n}_S,\chi)$ spanned by $E_1(1,\chi_R)$ and $E_1(1,\chi_S)$, and acts non-semisimply on this space, with generalised eigenvalue 1. More precisely, we have

$$\begin{aligned} U_{\mathfrak{p}}E_1(1,\chi_S) &= E_1(1,\chi_S), \\ U_{\mathfrak{p}}E_1(1,\chi_R) &= E_1(1,\chi_R) + E_1(1,\chi_S). \end{aligned}$$

The Hecke operators $U_{\mathfrak{q}}$ for $\mathfrak{q} \mid p$ and $\mathfrak{q} \neq \mathfrak{p}$ act on W_χ as the identity. It follows that e acts as the identity on W_χ, and in particular $eE_1(1,\chi_R) = E_1(1,\chi_R)$. This shows that $\nu_1(\mathscr{P}^o) = E_1(1,\chi_R)$. The calculation of $\nu_1(\mathscr{H})$ is a direct consequence of the fact that

$$\nu_1(\mathscr{E}(1,\chi)) = \nu_1(\mathscr{E}(\chi,1)) = E_1(1,\chi_S), \quad u_1 + v_1 = 1, \quad w_1 = 0.$$

Finally, the conditions satisfied by the Hecke operators $T_{(\eta,\psi)}$ in Lemma 2.9 and the fact that $E_1(1,\chi_S)$ is an eigenform show that $\nu_1(\mathscr{F})$ is a non-zero multiple of $E_1(1,\chi_S)$. This concludes the proof. $\square$

3.4 The weight $1+\varepsilon$ specialization

Let $\nu_{1+\varepsilon} : \Lambda_{(1)} \longrightarrow \tilde{E} = E[\varepsilon]/(\varepsilon^2)$ be the "weight $1+\varepsilon$ specialization" that was discussed in the introduction (cf. equation (31)). We now consider the form $H_{1+\varepsilon} := \nu_{1+\varepsilon}(\mathscr{H})$, and calculate its Fourier coefficients in terms of the homomorphism κ_{cyc} defined in (16). The Fourier coefficients of the form $\nu_1(\mathscr{H}) = E_1(1,\chi_S)$ are given, for primes $\mathfrak{q} \neq \mathfrak{p}$, by the sum of the two characters 1 and χ. The key observation is that the Fourier coefficients of the infinitesimal lift $H_{1+\varepsilon}$ are still given by the sum of two characters lifting 1 and χ.

Define these two characters $\psi_1, \psi_2 : G_F \to \tilde{E}^\times$ as follows. The character ψ_1 is unramified outside p and satisfies

$$\psi_1(\mathfrak{q}) = 1 + v_1 \kappa_{\text{cyc}}(\mathfrak{q})\varepsilon \qquad \text{for all } \mathfrak{q} \nmid p. \tag{89}$$

The character ψ_2 is unramified outside S and satisfies

$$\psi_2(\mathfrak{q}) = \chi(\mathfrak{q})(1 + u_1 \kappa_{\text{cyc}}(\mathfrak{q})\varepsilon) \qquad \text{for all } \mathfrak{q} \notin S. \tag{90}$$

As usual, we extend the definitions of ψ_1 and ψ_2 to totally multiplicative functions defined on the set of ideals of F by the following formulas, for prime $\mathfrak{q}$:

$$\begin{aligned} \psi_1(\mathfrak{q}) &= 1 \text{ if } \mathfrak{q} \mid p, \\ \psi_2(\mathfrak{q}) &= 0 \text{ if } \mathfrak{q} \in S. \end{aligned}$$

Proposition 3.6. *The Fourier coefficients of $H_{1+\varepsilon}$ satisfy $c(1, H_{1+\varepsilon}) = 1$ and, for each prime ideal $\mathfrak{q}$ of F, we have*

$$c(\mathfrak{q}, H_{1+\varepsilon}) = \psi_1(\mathfrak{q}) + \psi_2(\mathfrak{q}) \quad \text{if } \mathfrak{q} \neq \mathfrak{p} \tag{91}$$
$$c(\mathfrak{p}, H_{1+\varepsilon}) = 1 + w'(1)\varepsilon. \tag{92}$$

Furthermore, $H_{1+\varepsilon}$ is a simultaneous eigenform for the Hecke operators $T_\mathfrak{q}$ for $\mathfrak{q} \notin S$ and $U_\mathfrak{q}$ for $\mathfrak{q} \in S$ with eigenvalues given by (91) and (92).

Proof. We will only consider case 2, i.e. the case when $\mathfrak{p}$ is the only prime of F above p. (The analysis of case 1, where $v = 0$, can be treated by a similar but simpler calculation.) Write $\phi_{1+\varepsilon}(u) = u_1 + u_1'\varepsilon$, and likewise for v and w. Let us also write $E_{1+\varepsilon}(\psi, \eta) := \nu_{1+\varepsilon}(\mathscr{E}(\psi, \eta))$. Given an integral ideal $\mathfrak{m}$ of F, write $\mathfrak{m} = \mathfrak{b}\mathfrak{p}^t$ with $\mathfrak{p} \nmid \mathfrak{b}$. We note that

$$c(\mathfrak{m}, E_{1+\varepsilon}(1, \chi)) = \sum_{\mathfrak{r} \mid \mathfrak{b}} \chi(\mathfrak{r})(1 + \kappa_{\text{cyc}}(\mathfrak{r})\varepsilon), \tag{93}$$
$$c(\mathfrak{m}, E_{1+\varepsilon}(\chi, 1)) = \sum_{\mathfrak{r} \mid \mathfrak{b}} \chi(\mathfrak{r})(1 + \kappa_{\text{cyc}}(\mathfrak{b}/\mathfrak{r})\varepsilon). \tag{94}$$

It follows from the equations $w_1 = 0$ and $\nu_1(\mathscr{P}^o) = E_1(1, \chi)$ that

$$\begin{aligned} H_{1+\varepsilon} &= (u_1 E_{1+\varepsilon}(1, \chi) + v_1 E_{1+\varepsilon}(\chi, 1)) + \left(u_1' E_1(1, \chi\omega^0) + v_1' E_1(\chi, \omega^0) + w_1' E_1(1, \chi)\right)\varepsilon \\ &= (u_1 E_{1+\varepsilon}(1, \chi) + v_1 E_{1+\varepsilon}(\chi, 1)) + w_1' \left(E_1(1, \chi) - E_1(1, \chi_S)\right)\varepsilon. \end{aligned} \tag{95}$$

We have used the facts $E_1(1, \chi\omega^0) = E_1(\chi, \omega^0) = E_1(1, \chi_S)$ and $u_1' + v_1' + w_1' = 0$ in deriving (95). Using (93) and (94), the $\mathfrak{m}$th coefficient of the first term

$$u_1 E_{1+\varepsilon}(1, \chi) + v_1 E_{1+\varepsilon}(\chi, 1)$$

in (95) may be written:

$$\begin{aligned} & u_1\left(\sum_{\mathfrak{r}|\mathfrak{b}} \chi(\mathfrak{r})(1+\kappa_{\mathrm{cyc}}(\mathfrak{r})\varepsilon)\right) + v_1\left(\sum_{\mathfrak{r}|\mathfrak{b}} \chi(\mathfrak{r})(1+\kappa_{\mathrm{cyc}}(\mathfrak{b}/\mathfrak{r})\varepsilon)\right) \\ = & \sum_{\mathfrak{r}|\mathfrak{b}} \chi(\mathfrak{r})(1+u_1\kappa_{\mathrm{cyc}}(\mathfrak{r})\varepsilon + v_1\kappa_{\mathrm{cyc}}(\mathfrak{b}/\mathfrak{r})\varepsilon) \\ = & \sum_{\mathfrak{r}|\mathfrak{b}} \psi_1(\mathfrak{b}/\mathfrak{r})\psi_2(\mathfrak{r}). \end{aligned}$$

Here we have used the fact that $u_1 + v_1 = 1$, by (88).

The remaining term

$$w_1'\left(E_1(1,\chi) - E_1(1,\chi_S)\right)\varepsilon$$

in (95) has $\mathfrak{m}$th coefficient equal to

$$t \cdot w_1' \cdot (\sum_{\mathfrak{r}|\mathfrak{b}} \chi(\mathfrak{r}))\varepsilon,$$

so we obtain

$$\begin{aligned} c(\mathfrak{m}, H_{1+\varepsilon}) &= \sum_{\mathfrak{r}|\mathfrak{b}} \psi_1(\mathfrak{b}/\mathfrak{r})\psi_2(\mathfrak{r}) + tw_1' \sum_{\mathfrak{r}|\mathfrak{b}} \chi(\mathfrak{r})\varepsilon \\ &= \left(\sum_{\mathfrak{r}|\mathfrak{b}} \psi_1(\mathfrak{b}/\mathfrak{r})\psi_2(\mathfrak{r})\right) \times (1+w_1'\varepsilon)^t. \end{aligned} \tag{96}$$

Equations (91) and (92) follow immediately from (96). We leave to the reader the exercise, using (96) and the definition of the Hecke operators (see for instance equation (1.2.5) of [15]), that $H_{1+\varepsilon}$ is indeed an eigenvector for the Hecke operators with the specified eigenvalues.

□

The eigenform $H_{1+\varepsilon}$ determines a $\Lambda_{(1)}$-algebra homomorphism valued in the dual numbers:

$$\phi_{1+\varepsilon} : \tilde{\mathbf{T}} \otimes \Lambda_{(1)} \longrightarrow \tilde{E} = E[\varepsilon]/(\varepsilon^2),$$

defined by sending a Hecke operator to its eigenvalue on $H_{1+\varepsilon}$. The homomorphism $\phi_{1+\varepsilon}$ lifts the homomorphism

$$\phi_1 : \tilde{\mathbf{T}} \otimes \Lambda_{(1)} \longrightarrow E$$

describing the eigenvalues of the Hecke operators on the Eisenstein series $\nu_1(\mathscr{H}) = E_1(1, \chi_S)$. Geometrically, ϕ_1 corresponds to a point on the spectrum of $\tilde{\mathbf{T}}$, and $\phi_{1+\epsilon}$ describes a tangent vector at that point.

The fact that the cusp form $\mathscr{F}$ is obtained by applying a Hecke operator to $\mathscr{H}$, and that $\nu_1(\mathscr{F}) \neq 0$ (cf. Lemma 3.5), shows that $F_{1+\varepsilon}$ is (up to multiplication by an element of $\tilde{E}^\times$) a normalized cuspidal eigenform with Fourier coefficients in $\tilde{E}$ and with the same associated system of Hecke eigenvalues as $H_{1+\varepsilon}$. Hence $\phi_{1+\varepsilon}$ factors through the quotient $\mathbf{T}\otimes\Lambda_{(1)}$ of $\tilde{\mathbf{T}}\otimes\Lambda_{(1)}$.

Let $\mathbf{T}_{(1)}$ be the localization of $\mathbf{T}\otimes\Lambda_{(1)}$ at $\ker\phi_1$. We will view $\phi_{1+\varepsilon}$ as a homomorphism on $\mathbf{T}_{(1)}$, which is possible since $\phi_{1+\varepsilon}$ factors through the natural map $\mathbf{T}\otimes\Lambda_{(1)}\longrightarrow\mathbf{T}_{(1)}$. The following theorem summarizes the main results of this section.

Theorem 3.7. *Assume Leopoldt's conjecture for F, and in case 2 assume further that (11) holds. Let ψ_1,ψ_2 be as in (89) and (90). There exists a $\Lambda_{(1)}$-algebra homomorphism*

$$\phi_{1+\varepsilon}:\mathbf{T}_{(1)}\longrightarrow E[\varepsilon]/(\varepsilon^2)$$

satisfying

$$\begin{aligned}\phi_{1+\varepsilon}(T_{\mathfrak{q}}) &= \psi_1(\mathfrak{q})+\psi_2(\mathfrak{q}) \quad \textit{if } \mathfrak{q}\notin S; && (97)\\ \phi_{1+\varepsilon}(U_{\mathfrak{q}}) &= \psi_1(\mathfrak{q}) \quad \textit{if } \mathfrak{q}\in R, && (98)\\ \phi_{1+\varepsilon}(U_{\mathfrak{p}}) &= 1+u_1\mathscr{L}_{\mathrm{an}}(\chi)\varepsilon. && (99)\end{aligned}$$

Proof. Equations (97) and (98) simply restate (91). Equation (99) is a consequence of (92) and the identity (cf. equations (79) and (81))

$$w(k)=u(k)\mathscr{L}_{\mathrm{an}}(\chi,k),$$

which implies that $w_1'=u_1\mathscr{L}_{\mathrm{an}}(\chi)$. □

4 Galois representations

In this section, we parlay the homomorphism $\phi_{1+\varepsilon}$ of Theorem 3.7 into the construction of a cohomology class $\kappa\in H^1_{\mathfrak{p}}(F,E(\chi^{-1}))$ satisfying

$$\mathrm{res}_{\mathfrak{p}}\,\kappa=-\mathscr{L}_{\mathrm{an}}(\chi)\kappa_{\mathrm{nr}}+\kappa_{\mathrm{cyc}}. \qquad (100)$$

This class will be extracted from the Galois representations attached to the eigenforms in $\mathcal{S}^o(\mathfrak{n},\chi)$.

4.1 Representations attached to ordinary eigenforms

Recall that $\mathbf{T}_{(1)}$ is the localization of $\mathbf{T}\otimes\Lambda_{(1)}$ at $\ker\phi_1$. Let $\mathcal{F}_{(1)}$ be the total ring of fractions of the local ring $\mathbf{T}_{(1)}$. It is isomorphic to a product of fields:

$$\mathcal{F}_{(1)}=\mathcal{F}_1\times\cdots\times\mathcal{F}_t,$$

where each $\mathcal{F}_j$ is a finite extension of $\mathcal{F}_\Lambda$. Fix a factor $\mathcal{F}=\mathcal{F}_j$ in this decomposition. We will write $T_{\mathfrak{q}}$ and $U_{\mathfrak{q}}$ to denote the images of the corresponding Hecke operators in $\mathcal{F}$ under the natural projection $\mathcal{F}_{(1)}\longrightarrow\mathcal{F}$.

Recall the cyclotomic character $\epsilon_{\rm cyc} : G_F \longrightarrow \mathbf{Z}_p^\times$ that was introduced in equation (13) of the introduction, and satisfies

$$\epsilon_{\rm cyc}(\mathrm{Frob}_\mathfrak{q}) = \mathrm{N}\mathfrak{q}, \qquad \text{for all } \mathfrak{q} \nmid p.$$

The Λ-adic cyclotomic character

$$\underline{\epsilon}_{\rm cyc} : G_F \longrightarrow \Lambda^\times$$

is given by

$$\underline{\epsilon}_{\rm cyc}(\mathrm{Frob}_\mathfrak{q})(k) = \langle \mathrm{N}\mathfrak{q} \rangle^{k-1}.$$

The following key result is proved in [16] (cf. Theorems 2 and 4 of loc. cit.).

Theorem 4.1. *There is a continuous irreducible Galois representation*

$$\rho : G_F \to \mathbf{GL}_2(\mathcal{F})$$

satisfying

1. *The representation ρ is unramified at all primes $\mathfrak{q} \notin S$, and the characteristic polynomial of $\rho(\mathrm{Frob}_\mathfrak{q})$ is*
$$x^2 - T_\mathfrak{q} x + \chi(\mathfrak{q}) \langle \mathrm{N}\mathfrak{q} \rangle^{k-1}. \tag{101}$$
In particular, $\det \rho = \chi \underline{\epsilon}_{\rm cyc}$.

2. *The representation ρ is odd, i.e., the image of any complex conjugation in G_F under ρ has characteristic polynomial $x^2 - 1$.*

3. *For each $\mathfrak{q} \mid p$, let $G_\mathfrak{q} \subset G_S$ denote a decomposition group at $\mathfrak{q}$. Then*
$$\rho|_{G_\mathfrak{q}} \cong \begin{pmatrix} \chi \underline{\epsilon}_{\rm cyc} \eta_\mathfrak{q}^{-1} & * \\ 0 & \eta_\mathfrak{q} \end{pmatrix} \tag{102}$$
where $\eta_\mathfrak{q}$ is the unramified character of $G_\mathfrak{q}$ satisfying
$$\eta_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}) = U_\mathfrak{q}. \tag{103}$$

Let $V \cong \mathcal{F}^2$ be the representation space attached to ρ. For any choice of $\mathcal{F}$-basis of V there is an associated matrix representation

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}, \quad a, b, c, d : G_F \longrightarrow \mathcal{F}. \tag{104}$$

Let R denote the image of $\mathbf{T}_{(1)}$ under the projection $\mathcal{F}_{(1)} \longrightarrow \mathcal{F}$. The ring R is a quotient of $\mathbf{T}_{(1)}$ and hence is also a local ring having E as residue field. For $x \in R$, write $\overline{x} \in E$ for its reduction modulo the maximal ideal $\mathfrak{m}$ of R. Fix a choice of a complex conjugation $\delta \in G_F$. Since the representation ρ is totally odd, we may choose an $\mathcal{F}$-basis of V such that

$$\rho(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Assume once and for all that the basis of V has been chosen in this way.

Theorem 4.2. *The representation ρ satisfies the following properties.*

1. *For all $\sigma \in G_F$, the entries $a(\sigma)$ and $d(\sigma)$ belong to $R^\times$, and*

$$\overline{a}(\sigma) = 1, \qquad \overline{d}(\sigma) = \chi(\sigma).$$

2. *The matrix entry b does not vanish identically on the decomposition group $G_\mathfrak{p}$ at $\mathfrak{p}$.*

Proof. The traces of $\rho(\mathrm{Frob}_\mathfrak{q})$ are Hecke operators, and hence lie in R. By the Chebotarev density theorem, the same is true of $\mathrm{trace}(\rho(\sigma))$ for all $\sigma \in G_F$. The identities

$$a(\sigma) = \frac{1}{2}\left(\mathrm{trace}\,\rho(\sigma) + \mathrm{trace}\,\rho(\sigma\delta)\right) \tag{105}$$

and

$$d(\sigma) = \frac{1}{2}\left(\mathrm{trace}\,\rho(\sigma) - \mathrm{trace}\,\rho(\sigma\delta)\right) \tag{106}$$

imply that $a(\sigma), d(\sigma) \in R$. Furthermore, (101) combined with

$$\phi_1(T_\mathfrak{q}) = 1 + \chi(\mathfrak{q}) \text{ for } \mathfrak{q} \notin S$$

implies that $\overline{\mathrm{trace}\,\rho(\sigma)} = 1 + \chi$. Hence part 1 follows from (105) and (106). We now turn to the proof of part 2. Let B denote the R-submodule of $\mathcal{F}$ generated by the $b(\sigma)$ as σ ranges over G_F. A standard compactness argument shows that B is a finitely generated R-module. Since $d(\sigma)$ belongs to $R^\times$, the function $K(\sigma) := b(\sigma)/d(\sigma)$ also takes its values in B. Let

$$\overline{K} : G_F \longrightarrow \overline{B}, \qquad \overline{B} := B/\mathfrak{m}B$$

denote its mod $\mathfrak{m}$ reduction. A direct calculation using the multiplicativity of the representation ρ of equation (104) reveals that the function $\overline{K}$ is a continuous one-cocycle in $Z^1(G_F, \overline{B}(\chi^{-1}))$. Furthermore, part 3 of Theorem 4.1 gives, for each $\mathfrak{q} \mid p$, a change of basis matrix

$$\begin{pmatrix} A_\mathfrak{q} & B_\mathfrak{q} \\ C_\mathfrak{q} & D_\mathfrak{q} \end{pmatrix} \in \mathbf{GL}_2(\mathcal{F}) \tag{107}$$

satisfying

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} A_\mathfrak{q} & B_\mathfrak{q} \\ C_\mathfrak{q} & D_\mathfrak{q} \end{pmatrix} = \begin{pmatrix} A_\mathfrak{q} & B_\mathfrak{q} \\ C_\mathfrak{q} & D_\mathfrak{q} \end{pmatrix} \begin{pmatrix} \chi\underline{\epsilon}_{\mathrm{cyc}}\eta_\mathfrak{q}^{-1}(\sigma) & * \\ 0 & \eta_\mathfrak{q}(\sigma) \end{pmatrix} \tag{108}$$

for all $\sigma \in G_\mathfrak{q}$. Comparing the upper left-hand entries in (108) gives the identity

$$C_\mathfrak{q} b(\sigma) = A_\mathfrak{q}\left[\chi\underline{\epsilon}_{\mathrm{cyc}}\eta_\mathfrak{q}^{-1}(\sigma) - a(\sigma)\right] \tag{109}$$

for all $\sigma \in G_\mathfrak{q}$. Suppose now that $\mathfrak{q} \neq \mathfrak{p}$. Since $\chi \neq 1$ on $G_\mathfrak{q}$, there is a $\sigma_\mathfrak{q} \in G_\mathfrak{q}$ for which $\chi(\sigma_\mathfrak{q}) \neq 1$ and hence for which the expression in square braces on the right of (109), evaluated at $\sigma = \sigma_\mathfrak{q}$, is a unit in R. It follows that $C_\mathfrak{q} \neq 0$ and that the ratio $w_\mathfrak{q} := A_\mathfrak{q}/C_\mathfrak{q}$ belongs to B. Equation (109) can be rewritten as the equation in B:

$$K(\sigma) = \frac{\left[\chi \underline{\epsilon}_{\text{cyc}} \eta_\mathfrak{q}^{-1}(\sigma) - a(\sigma)\right]}{d(\sigma)} w_\mathfrak{q}, \quad \text{for all } \sigma \in G_\mathfrak{q}. \tag{110}$$

In order to reduce this equation modulo $\mathfrak{m}$, we note that

$$\nu_1(\underline{\epsilon}_{\text{cyc}}) = 1, \qquad \phi_1(\eta_\mathfrak{q}(\text{Frob}_\mathfrak{q})) = c(\mathfrak{q}, E_1(1, \chi_S)) = 1,$$

and hence $\phi_1(\eta_\mathfrak{q}(\sigma)) = 1$ for all $\sigma \in G_\mathfrak{q}$. Part 1 of Theorem 4.2 then yields

$$\overline{K}(\sigma) = (1 - \chi^{-1}(\sigma))\overline{w}_\mathfrak{q} \ \text{ for all } \sigma \in G_\mathfrak{q}. \tag{111}$$

It follows that $\overline{K}$ is locally trivial at all $\mathfrak{q} \mid p'$, and hence yields a class $[\overline{K}] \in H^1_\mathfrak{p}(F, \overline{B}(\chi^{-1}))$. Suppose that b vanishes on $G_\mathfrak{p}$. The same is then true of the cocycle $\overline{K}$, which is therefore trivial at $\mathfrak{p}$. Hence by Lemma 1.5, the cohomology class $[\overline{K}]$ is trivial. We may therefore write $\overline{K}$ as a coboundary:

$$\overline{K}(\sigma) = (1 - \chi^{-1}(\sigma))\theta, \tag{112}$$

for some $\theta \in \overline{B}$ and all $\sigma \in G_F$. Evaluating (112) at $\sigma = \delta$ shows that in fact $\theta = 0$, and hence $\overline{K} = 0$ as a cocycle on G_F. But the image of K generates the R-module B by definition, and hence $\overline{B} = 0$. Since B is a finitely generated R-module, Nakayama's Lemma implies that $B = 0$, contradicting the irreducibility of ρ. This contradiction proves part 2. □

4.2 Construction of a cocycle

In the previous section, we constructed, for each quotient $\mathcal{F}_j$ of $\mathcal{F}_{(1)}$, a specific Galois representation

$$\rho_j : G_F \longrightarrow \mathbf{GL}_2(\mathcal{F}_j).$$

The product of these representations yields a Galois representation

$$\rho_{(1)} : G_F \longrightarrow \mathbf{GL}_2(\mathcal{F}_{(1)}).$$

To lighten the notations we will write ρ instead of $\rho_{(1)}$ and use the symbols a, b, c and d to denote its matrix entries, with the understanding that in this section, these are now elements of $\mathcal{F}_{(1)} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_t$.

We now suppose we are given an $\Lambda_{(1)}$-algebra homomorphism $\phi_{1+\varepsilon} : \mathbf{T}_{(1)} \to \tilde{E}$ lifting the homomorphism ϕ_1 corresponding to the modular form $E_1(1, \chi_S)$. We suppose that $\phi_{1+\varepsilon}$ is given by the sum of two characters $\psi_1, \psi_2 : G_F \to \tilde{E}^\times$ lifting $1, \chi$, respectively, as in (97) and (98).

Lemma 4.3. *The Galois representation ρ satisfies the following properties:*

1. *For the chosen complex conjugation δ,*

$$\rho(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. *For all $\sigma \in G_F$, the entries $a(\sigma)$ and $d(\sigma)$ belong to $\mathbf{T}_{(1)}^{\times}$, and*

$$\phi_{1+\varepsilon} \circ a = \psi_1, \tag{113}$$
$$\phi_{1+\varepsilon} \circ d = \psi_2. \tag{114}$$

Proof. Part 1 follows directly from the definition of ρ. The proof of part 2 is identical to the proof of part 1 of Theorem 4.2, with R replaced by $\mathbf{T}_{(1)}$. □

The following is the main result of this section.

Theorem 4.4. *Let $\phi_{1+\varepsilon}$ be as above. Write $\psi_1 = 1 + \psi_1' \varepsilon$, and define $a_{\mathfrak{p}}' \in E$ by*

$$\phi_{1+\varepsilon}(U_{\mathfrak{p}}) = 1 + a_{\mathfrak{p}}' \varepsilon.$$

There exists a cohomology class $\kappa \in H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$ satisfying

$$\operatorname{res}_{\mathfrak{p}} \kappa = -a_{\mathfrak{p}}' \kappa_{\mathrm{nr}} + \kappa_{\mathrm{cyc}} - \operatorname{res}_{\mathfrak{p}}(\psi_1').$$

Proof. Consider the change of basis matrices

$$\begin{pmatrix} A_{\mathfrak{q}} & B_{\mathfrak{q}} \\ C_{\mathfrak{q}} & D_{\mathfrak{q}} \end{pmatrix}$$

with entries in $\mathcal{F}_{(1)}$ satisfying (108). Equation (109) with $\mathfrak{q} = \mathfrak{p}$, combined with part 2 of Theorem 4.2, shows that the projection of $A_{\mathfrak{p}}$ to each factor $\mathcal{F}_j$ is non-zero. Hence $A_{\mathfrak{p}}$ belongs to $\mathcal{F}_{(1)}^{\times}$. Let

$$\tilde{b}(\sigma) := \frac{C_{\mathfrak{p}}}{A_{\mathfrak{p}}} \times b(\sigma),$$

and write B for the $\mathbf{T}_{(1)}$-submodule of $\mathcal{F}_{(1)}$ generated by the $\tilde{b}(\sigma)$ as σ ranges over G_F. Since $d(\sigma)$ belongs to $\mathbf{T}_{(1)}^{\times}$, the function $K(\sigma) := \tilde{b}(\sigma)/d(\sigma)$ takes values in B as well. For each $\mathfrak{q} \mid p'$, the same reasoning as in the proof of Theorem 4.2 shows that the element $C_{\mathfrak{q}}$ belongs to $\mathcal{F}_{(1)}^{\times}$ (i.e. it is not a zero divisor), and that the element $x_{\mathfrak{q}} := \frac{C_{\mathfrak{p}}}{A_{\mathfrak{p}}} \times \frac{A_{\mathfrak{q}}}{C_{\mathfrak{q}}}$ belongs to B. The restriction of K to $G_{\mathfrak{q}}$ satisfies

$$K(\sigma) = \frac{\left[\chi \underline{\epsilon}_{\mathrm{cyc}} \eta_{\mathfrak{q}}^{-1}(\sigma) - a(\sigma)\right]}{d(\sigma)} x_{\mathfrak{q}}, \qquad \text{for all } \sigma \in G_{\mathfrak{q}}. \tag{115}$$

At the prime $\mathfrak{p}$, we have

$$K(\sigma) = \frac{\left[\underline{\epsilon}_{\mathrm{cyc}}(\sigma) \eta_{\mathfrak{p}}^{-1}(\sigma) - a(\sigma)\right]}{d(\sigma)}, \qquad \text{for all } \sigma \in G_{\mathfrak{p}}. \tag{116}$$

We now claim that the module B is contained in $\mathfrak{m} \subset \mathbf{T}_{(1)} \subset \mathcal{F}_{(1)}$. To see this, let $B^\sharp = (B+\mathfrak{m})/\mathfrak{m}$, and let $K^\sharp : G_F \longrightarrow B^\sharp$ be the function obtained by composing K with the natural surjection $B \longrightarrow B^\sharp$. The function

$$\overline{K}^\sharp : G_F \longrightarrow \overline{B}^\sharp := B^\sharp/\mathfrak{m}B^\sharp$$

obtained from $K^\sharp$ by reduction modulo the maximal ideal $\mathfrak{m}$ is a one-cocycle yielding a class

$$[\overline{K}^\sharp] \in H^1(F, \overline{B}^\sharp(\chi^{-1})).$$

Equation (116) shows that

$$K^\sharp(\sigma) = 0, \quad \text{and hence} \quad \overline{K}^\sharp(\sigma) = 0, \qquad \text{for all } \sigma \in G_{\mathfrak{p}}, \tag{117}$$

whereas equation (115) shows that $[\overline{K}^\sharp]$ is locally trivial at all $\mathfrak{q} \mid p'$. It follows from Lemma 1.3 that $[\overline{K}^\sharp] = 0$, and arguing with δ as in the proof of Theorem 4.2, we find that in fact $\overline{K}^\sharp = 0$ as a function. Since the $K^\sharp(\sigma)$ generate the finitely generated $\mathbf{T}_{(1)}$-module $B^\sharp$, we have $B^\sharp = \mathfrak{m}B^\sharp$ and hence $B^\sharp = 0$ by Nakayama's lemma. Thus $B \subset \mathfrak{m}$, as desired.

We can now complete the proof of Theorem 4.4. Since the function K takes values in $\mathfrak{m} \subset \mathbf{T}_{(1)}$, we may consider its composition with $\phi_{1+\varepsilon}$ and define a function $\kappa : G_F \to E$ by

$$\phi_{1+\varepsilon} \circ K(\sigma) = \kappa(\sigma)\varepsilon.$$

The function κ is a one-cocycle representing a class $[\kappa] \in H^1(F, E(\chi^{-1}))$. Since $x_{\mathfrak{q}} \in B \subset \mathfrak{m}$, (115) implies that $[\kappa]$ is locally trivial at all $\mathfrak{q} \mid p'$ and therefore belongs to $H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$. Finally, combining (116) with the equations

$$\begin{aligned} \phi_{1+\varepsilon} \circ \underline{\epsilon}_{\rm cyc} &= 1 + \kappa_{\rm cyc}\varepsilon, \\ \phi_{1+\varepsilon} \circ \eta_{\mathfrak{p}} &= 1 + a'_{\mathfrak{p}}\kappa_{\rm nr}\varepsilon, \\ \phi_{1+\varepsilon} \circ a &= 1 + \psi'_1\varepsilon, \\ \phi_{1+\varepsilon} \circ d &= \chi + \psi'_2\varepsilon, \end{aligned}$$

(the latter two arising in (113) and (114)) yields

$$\kappa(\sigma) = -a'_{\mathfrak{p}} \cdot \kappa_{\rm nr}(\sigma) + \kappa_{\rm cyc}(\sigma) - \psi'_1(\sigma)$$

for all $\sigma \in G_{\mathfrak{p}}$. Theorem 4.4 follows. □

Proof of Theorem 2. Applying Theorem 4.4 to the $\Lambda_{(1)}$-algebra homomorphism $\phi_{1+\varepsilon}$ constructed in Theorem 3.7, we find that $\psi'_1 = v_1\kappa_{\rm cyc}$ and $a'_{\mathfrak{p}} = u_1\mathscr{L}_{\rm an}(\chi)$, and hence obtain a class $\kappa \in H^1_{\mathfrak{p}}(F, E(\chi^{-1}))$ satisfying

$$\begin{aligned} \mathrm{res}_{\mathfrak{p}}\,\kappa &= -u_1\mathscr{L}_{\rm an}(\chi)\kappa_{\rm nr} + (1 - v_1)\kappa_{\rm cyc} \\ &= u_1(-\mathscr{L}_{\rm an}(\chi)\kappa_{\rm nr} + \kappa_{\rm cyc}). \end{aligned}$$

Recall that $u_1 \neq 0$ by (88). Replacing κ by κ/u_1, we obtain (100). This proves part 1 of Theorem 2, and part 2 for the character χ. The proof of part 2 for the character χ^{-1} follows from the observation that, when there is a unique prime of F above p, the roles of the matrix entries b and c can be interchanged in Theorems 4.2 and 4.4. This produces a cohomology class $\kappa \in H^1_{\mathfrak{p}}(F, E(\chi))$ satisfying

$$\mathrm{res}_{\mathfrak{p}}\,\kappa = -\mathscr{L}_{\mathrm{an}}(\chi^{-1})\kappa_{\mathrm{nr}} + \kappa_{\mathrm{cyc}}.$$

Theorem 2 follows. □

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