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Hilbert modular forms and the Gross-Stark conjecture. (English summary)
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Let F be a totally real field and χ an abelian totally odd character of F. In 1988, Gross stated a p-adic analogue of Stark's conjecture that relates the value of the derivative of the p-adic L-function associated to χ to the p-adic logarithm of a p-unit in the extension of F cut out by χ. In this paper the authors prove Gross's conjecture when F is a real quadratic field and χ a narrow ring class character. The main result also applies to general totally real fields for which Leopoldt's conjecture holds, assuming either that there are at least two primes above p in F, or that a certain condition relating the $\mathfrak{L}$-invariants of χ and χ^{-1} holds. This condition on $\mathfrak{L}$-invariants is always satisfied when χ is quadratic.

To be precise, let

- H be a finite cyclic extension of F cut out by χ,
- S be a finite set of places of F containing all the Archimedean places,
- $L_S(\chi, s)$ be the associated complex L-function,
- $\mathfrak{p}$ be a prime of F dividing p with $\chi(\mathfrak{p}) = 1$,
- $\mathfrak{P}$ be a prime of H lying above $\mathfrak{p}$,
- E be a finite extension of $\mathbf{Q}_p$ containing the values of χ,
- $\mathfrak{L}(\chi) \in E$ be a certain $\mathfrak{L}$-invariant defined by items from H but independent of these, and
- let ω be the p-adic Teichmüller character.

If S contains all the primes above p, Deligne and Ribet proved the existence of a continuous E-valued function $L_{S,p}(\chi\omega, s)$ of a variable $s \in \mathbf{Z}_p$ characterized by

$$L_{S,p}(\chi\omega, n) = L_S(\chi\omega^n, n)$$

for all integers $n \leq 0$.

In 1981, Gross proposed the following conjecture: For all characters χ of F and all $S = R \cup \mathfrak{p}$, one has

$$L'_{S,p}(\chi\omega, 0) = \mathfrak{L}(\chi) L_R(\chi, 0).$$

As explained in Section 1 of the paper, when $L_R(\chi, 0) = 0$, this conjecture follows from Wiles' proof of the Main Conjecture for totally real fields. In the sequel, one therefore supposes $L_R(\chi, 0) \neq 0$. In this setting, Gross's conjecture suggests defining an analytic $\mathfrak{L}$-invariant of χ by

$$\mathfrak{L}_{\mathrm{an}}(\chi) := \frac{L'_{S,p}(\chi\omega, 0)}{L_R(\chi, 0)} = \frac{d}{dk} \mathfrak{L}_{\mathrm{an}}(\chi, k)_{k=1},$$

where

$$\mathfrak{L}_{\mathrm{an}}(\chi, k) := \frac{-L'_{S,p}(\chi\omega, 1-k)}{L_R(\chi, 0)}.$$

Now, the main result of the paper is:

Theorem 2. Assume Leopoldt's conjecture holds for F.

(1) If there are at least two primes of F lying above $\mathfrak{p}$, then Gross's conjecture holds for all χ.
(2) If $\mathfrak{p}$ is the only prime of F lying above p, assume further that

$$\mathrm{ord}_{k=1}(\mathcal{L}_{\mathrm{an}}(\chi, k) + \mathcal{L}_{\mathrm{an}}(\chi^{-1}, k)) = \mathrm{ord}_{k=1}\mathcal{L}_{\mathrm{an}}(\chi^{-1}, k).$$

Then Gross's conjecture holds for both χ and χ^{-1}.

Theorem 2 leads to two unconditional results.

Corollary 5. Let F be a real quadratic field and let χ be a narrow ring class character of F. Then Gross's conjecture holds for χ.

Corollary 6. Let F be a totally real field satisfying Leopoldt's conjecture, and let χ be a narrow ray class character of F. Then Gross's conjecture holds for χ in either of the following two cases:

(1) there are at least two primes of F above the rational prime p, or
(2) the character χ is quadratic.

The proof of Theorem 2 starts by a cohomological interpretation of Gross's conjecture. In Section 1 of the paper, the problem of proving the theorem is transformed into the problem of constructing a global cohomology class κ in a certain subspace of the global cohomology group $H^1(F, E(\chi^{-1}))$ consisting of continuous classes whose restriction to the inertia subgroups $I_{\mathfrak{q}} \subset G_F$ are unramified at all primes $\mathfrak{q} \neq \mathfrak{p}$ of F.

As the authors state, the construction of κ borrows from techniques initiated by Ribet and extended by Wiles to prove the main conjecture of Iwasawa theory for totally real fields. Hence, Section 2 is devoted to Hilbert modular forms where the constant terms of certain Eisenstein series host the L-function values appearing in $L_{\mathrm{an}}(\chi, k)$.

Section 3, via p-adic interpolation, relates Hilbert modular forms to Λ-adic modular forms, Λ the Iwasawa algebra, a complete subring of the ring $\mathcal{C}(\mathbf{Z}_p, E)$ isomorphic to the power series ring $\mathcal{O}_E[[T]]$. Hecke operators acting on Λ-adic cusp forms and their Λ-algebra $\mathbf{T}$ play an essential role.

Section 4 completes the construction of the cohomology class κ. Here the key ingredient is a two-dimensional Galois representation

$$\rho\colon G_F \longrightarrow \mathrm{GL}_2(\mathcal{F}_{(1)})$$

where $\mathcal{F}_{(1)}$ is a total ring of fractions defined starting by $\mathbf{T}$.

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References

1. J. Coates and S. Lichtenbaum, On l-adic zeta functions, *Ann. of Math.* **98** (1973), 498–550. MR 0330107. Zbl 0279.12005. doi: 10.2307/1970916. MR0330107 (48 #8445)
2. P. Colmez, Réeidu en $s = 1$ des fonctione zêta p-adiques, *Invent. Math.* **91** (1988), 371–389. MR 0922806. Zbl 0651.12010. doi: 10.1007/BF01389373. MR0922806 (89d:11104)
3. P. Colmez, Fonctione zêta p-adiquee en $s = 0$, *J. Reine Angew. Math.* **467** (1995), 89–107. MR 1355923. Zbl 0864.11062. doi: 10.1515/crll.1995.467.89. MR1355923 (96i:11125)
4. P. Deligne and K. A. Ribet, Values of abelian L-functions at negative integers over to-

tally real fields, *Invent. Math.* **59** (1980), 227–286. MR 0579702. Zbl 0434. 12009. doi: 10.1007/BF01453237. MR0579702 (81m:12019)

5. B. Ferrero and R. Greenberg, On the behavior of p-adic L-functions at $s - 0$, *Invent. Math.* **50** (1978/79), 91–102. MR 0516606. Zbl 0441.12003. doi: 10.1007/BF01406470. MR0516606 (80f:12016)
6. R. Greenberg, On p-adic Artin L-functions, *Nagoya Math. J.* **89** (1983), 77–87. MR 0692344. Zbl 0513.12012. Available at http://projecteuclid.org/getRecord?id=euclid.nmj/1118787106. MR0692344 (85b:11104)
7. R. Greenberg, Trivial zeroe of p-adic L-functione, in *p-adic Monodromy and, the Birch and, Swinnerton-Dyer Conjecture* (Boston, MA, 1991), *Contemp. Math.* **165**, Amer. Math. Soc., Providence, RI, 1994, pp. 149–174. MR 1279608. Zbl 0838.11070. MR1279608 (95h:11063)
8. R. Greenberg, Introduction to Iwasawa theory for elliptic curvee, in *Arithmetic Algebraic Geometry* (Park City, UT, 1999), *IAS /Park City Math. Ser.* **9**, Amer. Math. Soc., Providence, RI, 2001, pp. 407–464. MR 1860044. Zbl 1002.11048. MR1860044 (2003a:11067)
9. B. H. Gross, p-adic L-series at $s - 0$, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28** (1981), 979–994 (1982). MR 0656068. Zbl 0507.12010. MR0656068 (84b:12022)
10. N. M. Katz, p-adic L-functions for CM fielde, *Invent. Math.* **49** (1978), 199–297. MR 0513095. Zbl 0417.12003. doi: 10.1007/BF01390187. MR0513095 (80h:10039)
11. T. Miyake, *Modular Forms*, Springer-Verlag, New York, 1989, translated from the Japanese by Yoehitaka Maeda. MR 1021004. Zbl 0701.11014. MR1021004 (90m:11062)
12. K. A. Ribet, A modular construction of unramified p-extensions of $\mathbf{Q}(\mu_p)$, *Invent. Math.* **34** (1976), 151–162. MR 0419403. Zbl 0338.12003. doi: 10.1007/BF01403065. MR0419403 (54 #7424)
13. G. Shimura, The special values of the zeta functions associated with Hilbert modular forms, *Duke Math. J.* **45** (1978), 637–679. MR 0507462. Zbl 0394.10015. Available at http://projecteuclid.org/getRecord?id=euclid.dmj/1077312955. MR0507462 (80a:10043)
14. C. L. Siegel,Über die Fourierschen Koeffizienten von Modulformen, *Nachr. Akad. Wise. Göttingen Math.-Phys. Kl. II* **1970** (1970), 15–56. MR 0285488. Zbl 0225.10031. MR0285488 (44 #2706)
15. J. Tate, *Les Conjectures de Stark sur les Fonctions L d'Artin en $s = 0$*, Lecture notes edited by Dominique Bernardi and Norbert Schappacher, *Progr. Math.* **47**, Birkhäuser Boston Inc., Boston, MA, 1984. MR 0782485. Zbl 0545.12009. MR0782485 (86e:11112)
16. A. Wiles, On p-adic representations for totally real fields, *Ann. of Math.* **123** (1986), 407–456. MR 0840720. Zbl 0613.12013. doi: 10.2307/1971332. MR0840720 (87g:11142)
17. A. Wiles, On ordinary λ-adic representations associated to modular forms, *Invent. Math.* **94** (1988), 529–573. MR 0969243. Zbl 0664.10013. doi: 10.1007/BF01394275. MR0969243 (89j:11051)
18. A. Wiles, The Iwasawa conjecture for totally real fielda, *Ann. of Math.* **131** (1990), 493–540. MR 1053488. Zbl 0719.11071. doi: 10.2307/1971468. MR1053488 (91i:11163)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.