

Iwasawa Theory of Elliptic Curves at Supersingular Primes over \mathbf{Z}_p -extensions of Number Fields

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Abstract

In this paper, we make a study of the Iwasawa theory of an elliptic curve at a supersingular prime p along an arbitrary \mathbf{Z}_p -extension of a number field K in the case when p splits completely in K . Generalizing work of Kobayashi [9] and Perrin-Riou [17], we define restricted Selmer groups and λ^\pm , μ^\pm -invariants; we then derive asymptotic formulas describing the growth of the Selmer group in terms of these invariants. To be able to work with non-cyclotomic \mathbf{Z}_p -extensions, a new local result is proven that gives a complete description of the formal group of an elliptic curve at a supersingular prime along any ramified \mathbf{Z}_p -extension of \mathbf{Q}_p .

1 Introduction

Over the last few years, much light has been shed on the subject of Iwasawa theory of elliptic curves at supersingular primes. In [10] and [17], asymptotic formulas for the size of $\text{III}(E/\mathbf{Q}_n)[p^\infty]$ have been established where \mathbf{Q}_n runs through the cyclotomic \mathbf{Z}_p -extension of \mathbf{Q} . In [9] and [18], a theory of algebraic and analytic p -adic L -functions is formed that closely parallels the case of ordinary reduction. The methods of all of the above papers depend heavily upon varying the fields considered in the cyclotomic direction. The purpose of this paper is to extend some of these results to a more general collection of \mathbf{Z}_p -extensions.

The essential difference in Iwasawa theory between the ordinary and the supersingular case is that, in the later case, the Galois theory of Selmer groups is badly behaved. Namely, if K_∞/K is a \mathbf{Z}_p -extension with layers K_n and E/K is an elliptic curve supersingular at some prime over p , then the Selmer group of E over K_n is much smaller than the $\text{Gal}(K_\infty/K_n)$ -invariants of the Selmer group of E over K_∞ . (In the case of ordinary reduction, these two groups are nearly the same by Mazur's control theorem.) The reason descent fails in the supersingular case boils down to the fact that the trace map on \widehat{E} (the formal group of E/\mathbf{Q}_p) is not surjective along a ramified \mathbf{Z}_p -extension. Following [14],

we make a careful study of how the trace map affects the Galois theory and we propose an analogous “control theorem” that takes into account the formal group of E (see Theorem 3.1). In the end, this setup allows one to convert the local information of \widehat{E} into global information about the Selmer group of E . These considerations are carried out in section 3.

In [9], a complete description of the Galois module structure of $\widehat{E}(k_n)$ is given in terms of generators and relations where k_n runs through the local cyclotomic \mathbf{Z}_p -extension of \mathbf{Q}_p . The new local result of this paper is a generalization of the above result to any ramified \mathbf{Z}_p -extension of \mathbf{Q}_p . Namely, if L_∞/\mathbf{Q}_p is a ramified \mathbf{Z}_p -extension with layers L_n , we produce points $d_n \in \widehat{E}(L_n)$ such that $\text{Tr}_{n-1}^n(d_n) = -d_{n-2}$ for $n \geq 2$ where $\text{Tr}_{n-1}^n : \widehat{E}(L_n) \rightarrow \widehat{E}(L_{n-1})$ is the trace map. Furthermore, d_n and d_{n-1} generate $\widehat{E}(L_n)$ over $\mathbf{Z}_p[\text{Gal}(L_n/\mathbf{Q}_p)]$ (see Theorem 4.5). From this result, we can completely describe the kernel and cokernel of the trace map. This local analysis is done in section 4. Note that the above analysis of \widehat{E} not only gives generators and relations, but the generators satisfy a compatibility as the level varies. It is precisely this compatibility that allows Iwasawa theory in the supersingular case to retain the flavor of the ordinary case.

To be able to apply these local results, we are obliged to work with number fields K for which p splits completely (since the local result assumes that we are working over \mathbf{Q}_p). For such K and p , we analyze arbitrary \mathbf{Z}_p -extensions of K . Following [14], we produce algebraic p -adic L -functions and then using the ideas of [9] and [18] we form plus/minus p -adic L -functions that actually lie in the Iwasawa algebra (assuming $a_p = 0$). Attached to these L -functions, we can associate plus/minus μ and λ -invariants.

In section 5, we analyze the case where these L -functions are units (i.e. when all the μ and λ -invariants are zero). In terms of E , this is the case when $E(K)/pE(K) = 0$, $\text{III}(E/K)[p] = 0$ and $p \nmid \text{Tam}(E/K)$; here $\text{Tam}(E/K)$ represents the Tamagawa factor of E over K . (Note that by the Birch and Swinnerton-Dyer conjecture, these hypotheses are equivalent to $\frac{L(E/K,1)}{\Omega_{E/K}}$ being a p -adic unit.) Under these strict global hypotheses, we prove that $\widehat{E}(K_n)$ and $\text{III}(E/K_n)[p^\infty]$ are finite for all n . Furthermore, we describe precisely the Galois structure of $\text{III}(E/K_n)[p^\infty]$ and, in particular, produce precise formulas for its size.

When $K = \mathbf{Q}$ and $\frac{L(E/\mathbf{Q},1)}{\Omega_{E/\mathbf{Q}}}$ is a p -adic unit, using Kato’s Euler system we can verify our algebraic hypotheses and we recover the main result of [10] (see Corollary 5.10). When K is an imaginary quadratic extension of \mathbf{Q} where p splits and $\frac{L(E/K,1)}{\Omega_{E/K}}$ is a p -adic unit, we can again verify our algebraic hypotheses via Kato’s result and produce exact descriptions of the the size and structure of $\text{III}(E/K_n)[p^\infty]$ (see Corollary 5.11).

In section 6, we give two different constructions of these plus/minus algebraic p -adic L -functions. Namely, we follow [14] and use the points $\{d_n\}$ to produce p -adic power series. Alternatively, we use the methods of [9] to produce restricted Selmer groups (which behave more like Selmer groups at ordinary

primes). These two approaches are related in that the characteristic power series of the restricted Selmer groups agree with the power series constructed (see Proposition 6.9).

Finally, in section 7, we study the arithmetic of E along the extension K_∞/K . When the coranks of the Selmer groups grow without bound along this extension, the algebraic p -adic L -functions vanish and the restricted Selmer groups are not cotorsion (over the Iwasawa algebra). In this case, the coranks of these restricted Selmer groups control the rate of growth of the coranks of the Selmer groups at each finite level (see Proposition 7.1). On the other hand, when these coranks remain bounded, we prove that these L -functions are non-zero and the restricted Selmer groups are indeed cotorsion. In this case, we produce asymptotic formulas for the growth of these Selmer group in terms of the Iwasawa invariants of the plus/minus p -adic L -functions as in [17] (see Theorem 7.14).

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2 Preliminaries

Let E/\mathbf{Q} be an elliptic curve and p an odd supersingular prime for E . Let K/\mathbf{Q} be a finite extension and K_∞/K a \mathbf{Z}_p -extension with layers K_n . Denote by Λ the Iwasawa algebra $\mathbf{Z}_p[[\text{Gal}(K_\infty/K)]]$ and let $\Gamma_n = \text{Gal}(K_\infty/K_n)$. We will impose the following hypothesis on the splitting type of the prime p in K_∞ .

- **Hypothesis S** (for splitting type): The prime p splits completely in K into $d = [K : \mathbf{Q}]$ distinct primes, say $\mathfrak{p}_1, \dots, \mathfrak{p}_d$. Also, each \mathfrak{p}_i is totally ramified in K_∞ .

Lemma 2.1. *Hypothesis (S) implies that $E(K_{\mathfrak{p}_i})[p] = 0$ and $E(K_\infty)[p] = 0$.*

Proof. We have an exact sequence

$$0 \longrightarrow E_1(\mathbf{Q}_p) \longrightarrow E(\mathbf{Q}_p) \longrightarrow \tilde{E}(\mathbf{F}_p) \longrightarrow 0 \quad (1)$$

where \tilde{E} denotes the reduction of $E \bmod p$ and where E_1 is defined by the above sequence. Since p is supersingular, $\tilde{E}(\mathbf{F}_p)$ has no p -torsion (see [22, V. Theorem 3.1]). Furthermore, since $E_1(\mathbf{Q}_p) \cong \widehat{E}(\mathbf{Q}_p)$, we have $E_1(\mathbf{Q}_p)$ has no p -torsion (see [22, VII. Proposition 2.2 and IV. Theorem 6.1]). Hence, $E(\mathbf{Q}_p)[p] = 0$. Now since p splits completely in K , we have that $K_{\mathfrak{p}_i} \cong \mathbf{Q}_p$ and $E(K_{\mathfrak{p}_i})[p] = 0$.

For the second part, if $E(K_\infty)[p] \neq 0$, then $E(K)[p] = E(K_\infty)[p]^\Gamma \neq 0$ since $\Gamma = \text{Gal}(K_\infty/K)$ is pro- p . However, $E(K)[p] \subseteq E(K_{\mathfrak{p}_i})[p] = 0$. \square

2.1 Selmer groups

For L an algebraic extension of \mathbf{Q} and v a prime of L , define

$$\mathcal{H}_E(L_v) = \frac{H^1(L_v, E[p^\infty])}{E(L_v) \otimes \mathbf{Q}_p/\mathbf{Z}_p} \quad \text{and} \quad \mathcal{P}_E(L) = \prod_v \mathcal{H}_E(L_v)$$

where the product is taken over all primes of L . (Here L_v is a union of completions of finite extensions of \mathbf{Q} in L .) Then the Selmer group of $E[p^\infty]$ is defined as

$$\text{Sel}(E[p^\infty]/L) = \ker(H^1(L, E[p^\infty]) \longrightarrow \mathcal{P}_E(L)).$$

We then have

$$0 \longrightarrow E(L) \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow \text{Sel}(E[p^\infty]/L) \longrightarrow \text{III}(E/L)[p^\infty] \longrightarrow 0$$

where $\text{III}(E/L)$ denotes the Tate-Shafarevich group. The Selmer group of $T_p E$ (the Tate module of E) is defined similarly (the cocycles should locally lie in $E(L_v) \otimes \mathbf{Z}_p$) and will be denoted by $\text{Sel}(T_p E/L)$.

We will use the following abbreviations:

$$S_n = \text{Sel}(E[p^\infty]/K_n), \quad S_\infty = \text{Sel}(E[p^\infty]/K_\infty),$$

$$S_n(T) = \text{Sel}(T_p E/K_n), \quad \mathfrak{X}_n = (S_n)^\wedge \quad \text{and} \quad \mathfrak{X}_\infty = (S_\infty)^\wedge$$

where $Y^\wedge = \text{Hom}(Y, \mathbf{Q}_p/\mathbf{Z}_p)$.

2.2 Local duality

Theorem 2.2 (Tate Local Duality). *Let v be a finite place of K . There exists a perfect pairing*

$$H^1(K_v, E[p^\infty]) \times H^1(K_v, T_p E) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p$$

induced by cup-product. Furthermore, under this pairing $E(K_v) \otimes \mathbf{Q}_p/\mathbf{Z}_p$ is the exact annihilator of $E(K_v) \otimes \mathbf{Z}_p$, inducing an isomorphism

$$\mathcal{H}_E(K_v)^\wedge \cong E(K_v) \otimes \mathbf{Z}_p. \tag{2}$$

Proof. See [23, Theorem 2.1]. □

We can use local duality to analyze the local factors $\mathcal{H}_E(K_v)$ appearing in the definition of the Selmer group.

Lemma 2.3.

1. *If $v \nmid p$, then $\mathcal{H}_E(K_v)$ is finite.*
2. *If $\mathfrak{p}|p$, then $\mathcal{H}_E(K_{\mathfrak{p}})^\wedge \cong \widehat{E}(K_{\mathfrak{p}})$ assuming hypothesis (S).*

Proof. We have that $E(K_v) \cong \mathbf{Z}_l^{[K_v:\mathbf{Q}_p]} \times T$ where $v \mid l$ and T is a finite group (see [22, VII. Proposition 6.3]). Hence by (2), $\mathcal{H}_E(K_v) \cong (T \otimes \mathbf{Z}_p)^\wedge$ if $l \neq p$ and is therefore finite. For $\mathfrak{p}|p$,

$$\mathcal{H}_E(K_{\mathfrak{p}})^\wedge \cong E(K_{\mathfrak{p}}) \otimes \mathbf{Z}_p \cong E_1(K_{\mathfrak{p}}) \otimes \mathbf{Z}_p \cong \widehat{E}(K_{\mathfrak{p}})$$

by (1) since p is supersingular and $K_{\mathfrak{p}} \cong \mathbf{Q}_p$. □

2.3 Global duality

Let Σ be a finite set of primes of L containing p , the infinite primes and all primes of bad reduction for E and let K_Σ be the maximal extension of K that is unramified outside of Σ . We have two exact sequences

$$0 \longrightarrow S_n \longrightarrow H_n \xrightarrow{\gamma_n} \bigoplus_{v \in \Sigma} \mathcal{H}_E(K_{n,v}) \quad (3)$$

and

$$0 \longrightarrow S_{n,\Sigma}(T) \longrightarrow S_n(T) \longrightarrow \bigoplus_{v \in \Sigma} E(K_v) \otimes \mathbf{Z}_p \quad (4)$$

where $H_n = H^1(K_\Sigma/K_n, E[p^\infty])$ and where $S_{n,\Sigma}(T)$ is defined by the second sequence. By Tate local duality, $E(K_v) \otimes \mathbf{Z}_p$ is dual to $\mathcal{H}_E(K_{n,v})$. Global duality asserts that these two sequences splice into a five term exact sequence.

Theorem 2.4 (Global duality). *The sequence*

$$0 \longrightarrow S_n \longrightarrow H_n \xrightarrow{\gamma_n} \bigoplus_{v \in \Sigma} \mathcal{H}_E(K_{n,v}) \longrightarrow S_n(T)^\wedge \longrightarrow S_{n,\Sigma}(T)^\wedge \longrightarrow 0$$

is exact where the first two maps come from (3) and the last two maps come from (4) and Tate local duality.

Proof. For a statement of global duality in this form see [20, Section 1.7]. \square

3 A control theorem in the supersingular case

When p is an ordinary prime for E , Mazur proved that the natural map of restriction between S_n and $S_\infty^{\Gamma_n}$ has finite kernel and cokernel of size bounded independent of n (see [12]). A theorem of this form, that compares S_n to $S_\infty^{\Gamma_n}$ is often called a control theorem. A key ingredient needed for this result is that the trace map on the formal group of E is surjective along a ramified \mathbf{Z}_p -extension.

In the supersingular case, the trace fails to be surjective (see [11]). In fact, the \mathbf{Z}_p -corank of $\text{coker}(S_n \rightarrow S_\infty^{\Gamma_n})$ grows without bound. In this section, we will produce an analogous control theorem that describes this cokernel in terms of the formal group of E . Throughout this section, we will be assuming (S).

Let $s_n : S_n \rightarrow S_\infty^{\Gamma_n}$ and $r_{n,v} : \mathcal{H}_E(K_{n,v}) \rightarrow \mathcal{H}_E(K_{\infty,v'})$ denote the natural restriction maps with v' some prime of K_∞ over v . The following theorem can be thought of as a control theorem in the supersingular case.

Theorem 3.1. *We have a four term exact sequence*

$$0 \longrightarrow \frac{S_n(T)}{S_{n,\Sigma}(T)} \longrightarrow \widehat{E}(K_{n,p}) \times B_n \longrightarrow (\mathfrak{X}_\infty)_{\Gamma_n} \xrightarrow{x_n} \mathfrak{X}_n \longrightarrow 0$$

where $x_n = (s_n)^\wedge$, $\widehat{E}(K_{n,p}) = \bigoplus_{j=1}^d \widehat{E}(K_{n,p_j})$ and B_n is a finite group whose size is bounded by the p -part of $\text{Tam}(E/K_n)$.

To prove this theorem, we will need to control the kernel and cokernel of s_n . We follow the methods of [3] and [4] and direct the reader to these articles for more details.

Proposition 3.2. *We have*

1. $\ker(s_n) = 0$
2. $\operatorname{coker}(s_n) \cong \operatorname{im}(\gamma_n) \cap (\bigoplus_{v \in \Sigma} \ker(r_{n,v}))$

where γ_n is defined in (3).

Proof. This proposition follows from applying the snake lemma to the diagram defining S_n and S_∞ . See [3, Chapter 4] especially Lemma 4.2 and 4.3 for details. \square

The following proposition describes $\ker(r_{n,v})$. The case of primes dividing p behaves quite differently from primes not over p .

Proposition 3.3. *We have*

1. For $v \nmid p$, $\ker(r_{n,v})$ is finite. If v splits completely in K_∞ , then $\ker(r_{n,v}) = 0$; otherwise it has size equal to $\operatorname{Tam}(E/K_{n,v})$ up to a p -adic unit.
2. For $\mathfrak{p}|p$, $\ker(r_{n,\mathfrak{p}}) = \mathcal{H}_E(K_{n,\mathfrak{p}})$.

Proof. For part (1), see the comments after Lemma 3.3 in [4]. For part (2), we have

$$\mathcal{H}_E(K_{\infty,\mathfrak{p}}) = \varprojlim_n \mathcal{H}_E(K_{n,\mathfrak{p}}) = \left(\varprojlim_n \widehat{E}(K_{n,\mathfrak{p}}) \right)^\wedge$$

where the last inverse limit is taken with respect to the trace map. However, there are no universal norms for \widehat{E} along the ramified \mathbf{Z}_p -extension $K_{\infty,\mathfrak{p}}/K_{\mathfrak{p}}$ since p is supersingular (see [11]). Therefore, $\mathcal{H}_E(K_{\infty,\mathfrak{p}}) = 0$ and $\ker(r_{n,\mathfrak{p}}) = \mathcal{H}_E(K_{n,\mathfrak{p}})$. \square

Remark 3.4. The fact that $\ker(r_{n,\mathfrak{p}})$ equals all of $\mathcal{H}_E(K_{n,\mathfrak{p}})$ is the essential difference between the ordinary case and the supersingular case and is the reason why the cokernel of s_n grows without bound.

Proof of Theorem 3.1. To control $\operatorname{coker}(s_n)$ we will need to understand how $\operatorname{im}(\gamma_n)$ relates to $\bigoplus_{v \in \Sigma} \ker(r_{n,v})$. To ease notation, let $\mathcal{H}^p = \bigoplus_{v \nmid p} \mathcal{H}_E(K_{n,v})$ and $\mathcal{H}_p = \bigoplus_{\mathfrak{p}|p} \mathcal{H}_E(K_{n,\mathfrak{p}})$. By global duality

$$\operatorname{im}(\gamma_n) = \ker(\mathcal{H}^p \times \mathcal{H}_p \longrightarrow S_n(T)^\wedge).$$

For $v \nmid p$, the image of $\mathcal{H}_E(K_{n,v})$ in $S_n(T)^\wedge$ is zero (the former is a finite group by Lemma 2.3 and the latter is a free module). Hence, we can write $\operatorname{im}(\gamma_n) = \mathcal{H}^p \times A$ with $A \subseteq \mathcal{H}_p$ and applying global duality again yields

$$\frac{\mathcal{H}_p}{A} = \left(\frac{S_n(T)}{S_{n,\Sigma}(T)} \right)^\wedge. \quad (5)$$

By Proposition 3.3, $\bigoplus_{\mathfrak{p}|p} \ker(r_{n,\mathfrak{p}}) = \mathcal{H}_p$ and hence

$$\mathrm{im}(\gamma_n) \cap (\bigoplus_{v \in \Sigma} \ker(r_{n,v})) \cong (\bigoplus_{v \nmid p} \ker(r_{n,v})) \times A.$$

Therefore, by Proposition 3.2 and (5), we have

$$0 \longrightarrow \mathrm{coker}(s_n) \longrightarrow (\bigoplus_{v \nmid p} \ker(r_{n,v})) \times \mathcal{H}_p \longrightarrow \left(\frac{S_n(T)}{S_{n,\Sigma}(T)} \right)^\wedge \longrightarrow 0. \quad (6)$$

By Lemma 2.3, $(\mathcal{H}_p)^\wedge \cong \widehat{E}(K_{n,p})$ and by Proposition 3.3, $\# \ker(r_{n,v})$ is bounded by the p -part of $\mathrm{Tam}(E/K_{n,v})$. Therefore, dualizing (6) yields the theorem. \square

4 Structure of some formal groups

4.1 Lubin-Tate formal groups

Let $p > 2$ be a prime and $\{L_n\}_{n \geq 0}$ with $\mathbf{Q}_p = L_0 \subset L_1 \subset L_2 \dots \subset L_\infty = \bigcup_n L_n$ be a tower of fields such that L_∞ is a totally ramified \mathbf{Z}_p -extension of \mathbf{Q}_p . Let $k_{n+1} := L_n[\mu_p]$ for $n \geq 0$ and $k_\infty = \bigcup_n k_n = L_\infty[\mu_p]$. Here, if M is a field by $M[\mu_p]$ we mean the extension of M obtained by adjoining to M the p -th roots of unity in some fixed algebraic closure of M . Then k_∞ is a \mathbf{Z}_p^\times -extension of \mathbf{Q}_p and the group of its universal norms is generated by a uniformizer of \mathbf{Z}_p , say π , such that $\mathrm{ord}_p\left(\frac{\pi}{p} - 1\right) > 0$. Now we would like to carefully choose a Lubin-Tate formal group (by choosing a ‘‘lift of Frobenius’’ corresponding to π) whose π^n -division points generate k_n over \mathbf{Q}_p .

Namely, let us define

$$f(X) := \pi X + \sum_{i=2}^p \frac{p(p-1) \cdots (p-i+1)}{i!} X^i \in \mathbf{Z}_p[[X]].$$

Then $f(X)$ is a lift of Frobenius corresponding to π , that is $f(X) = \pi X \pmod{\deg 2}$ and $f(X) = X^p \pmod{p}$ and moreover it satisfies the properties:

1. $f(X) = (X+1)^p - 1 \pmod{p^2}$
2. the coefficient of X^{p-1} is p .

We call this a *good lift* of Frobenius.

Lemma 4.1. *For π as above and for a good lift of Frobenius $f(X)$, let us denote by $F_f(X, Y)$ the corresponding formal group law. We have*

$$F_f(X, Y) = X + Y + XY \pmod{p} \quad \text{and} \quad [a]_f(X) = (X+1)^a - 1 \pmod{p}$$

for all $a \in \mathbf{Z}_p$.

Proof. Let us write $f(X) = (X+1)^p - 1 + p^2g(X)$ where $g(X) \in \mathbf{Z}_p[[X]]$. Then $F_f(X, Y)$ (respectively $[a]_f(X)$) is the unique power series with coefficients in \mathbf{Z}_p such that $F_f(X, Y) = X + Y \pmod{\deg 2}$ and $f(F_f(X, Y)) = F_f(f(X), f(Y))$ (resp. such that $[a]_f(X) = aX \pmod{\deg 2}$ and $f([a]_f(X)) = [a]_f(f(X))$).

Writing the identity for F_f we get:

$$(F_f(X, Y) + 1)^p - 1 + p^2g(F_f(X, Y)) = F_f((X+1)^p - 1, (Y+1)^p - 1) + p^2G(X, Y)$$

for some $G(X, Y) \in \mathbf{Z}_p[[X, Y]]$. Therefore $F_f(X, Y)$ satisfies the identity

$$(F_f(X, Y) + 1)^p - 1 = F_f((X+1)^p - 1, (Y+1)^p - 1) \pmod{p^2}. \quad (7)$$

Let

$$F_f(X, Y) = X + Y + \sum_{i, j \geq 1} a_{ij} X^i Y^j \quad (8)$$

with $a_{ij} \in \mathbf{Z}_p$. One checks that the identity (7) uniquely determines each a_{ij} modulo p . Since the power series $X + Y + XY$ satisfies (7), we must then have that $F_f(X, Y) = X + Y + XY \pmod{p}$. The proof for $[a]_f(X)$ is similar. \square

Let us fix for the rest of this section $f(X)$ and $F_f(X, Y)$ as in Lemma 4.1 and let $[i]_f(X) = \sum_{j=1}^{\infty} a_j(i) X^j$, for $i \geq 1$.

Corollary 4.2. *For any $m \geq 1$, we have that the determinant $(a_j(i))_{1 \leq i \leq m, 1 \leq j \leq m}$ is in \mathbf{Z}_p^\times .*

Proof. From Lemma 4.1, we see that $(a_j(i))$ is a lower triangular matrix modulo p with ones along the diagonal. Hence, $\det(a_i(j)) \in \mathbf{Z}_p^\times$. \square

Let us denote by \mathcal{O}_n the ring of integers in k_n and by M_n its maximal ideal. For every n , we have $k_n = \mathbf{Q}_p(F_f[\pi^n])$ where $F_f[\pi^n]$ denotes the π^n -torsion of the formal group F_f .

Corollary 4.3. *Let $\beta \in F_f[\pi^n]$ and fix some $m \geq 1$. Then we can find a linear combination with coefficients in \mathbf{Z}_p of*

$$[1]_f(\beta), [2]_f(\beta), \dots, [m]_f(\beta)$$

which has the form $\beta^m + \beta^{m+1}V$ for some $V \in \mathcal{O}_n$.

Proof. Apply Corollary 4.2. \square

For every $n \in \mathbf{Z}^{\geq 1}$ we denote by $G_n(f)$ the \mathbf{Z}_p -submodule of M_n generated by $F_f[\pi^n]$. The main result of this section is the following proposition.

Proposition 4.4.

1. *We have $G_n(f) = M_n$ for all $n \geq 1$.*
2. *Each $\beta \in F_f[\pi^n] - F_f[\pi^{n-1}]$ generates M_n/M_{n-1} as a $\mathbf{Z}_p[\text{Gal}(k_n/\mathbf{Q}_p)]$ -module.*

Proof. As $F_f[\pi^{n-1}] \subset M_{n-1}$ and as $\text{Gal}(k_n/\mathbf{Q}_p)(\beta) = F_f[\pi^n] - F_f[\pi^{n-1}]$, part (1) implies part (2).

To prove part (1), it suffices to show that $G_n(f)$ contains elements of valuation $\frac{b}{p^n - p^{n-1}}$ for all $b \geq 1$. But this follows immediately from Corollary 4.3. \square

4.2 Formal groups of elliptic curves with supersingular reduction

Let E/\mathbf{Q}_p be an elliptic curve with supersingular reduction and suppose that $a_p = 0$. Let us denote, as in the previous sections, by \widehat{E} the formal group of E , i.e. the formal scheme over \mathbf{Z}_p which is the formal completion of the Néron model of E at the identity of its special fiber. Let L_∞/\mathbf{Q}_p be a ramified \mathbf{Z}_p -extension with layers L_n . We denote by $\text{Tr}_{n-1}^n : \widehat{E}(L_n) \rightarrow \widehat{E}(L_{n-1})$ the trace with respect to the group-law $\widehat{E}(X, Y)$. Then the following is the main result of this section.

Theorem 4.5. *For $n \geq 0$ there exists $d_n \in \widehat{E}(L_n)$ such that*

1. $\text{Tr}_{n-1}^n d_n = -d_{n-2}$
2. $\text{Tr}_0^1 d_1 = u \cdot d_0$ with $u \in \mathbf{Z}_p^\times$.
3. For $n \geq 1$, $\widehat{E}(L_n)$ is generated by d_n and d_{n-1} as a $\mathbf{Z}_p[\text{Gal}(L_n/\mathbf{Q}_p)]$ -module. Also, d_0 generates $\widehat{E}(\mathbf{Q}_p)$.

The proof of this theorem will fill the rest of this section. Let us consider the \mathbf{Z}_p^\times -extension k_∞ attached to L_∞ as in the section 4.1 and denote by π the generator of the group of universal norms of the extension k_∞/\mathbf{Q}_p which has positive valuation. We will first construct a sequence of points $c_n \in \widehat{E}(k_n)$ which satisfy the same trace relations and then use these points to construct the points of Theorem 4.5.

We will use Honda-theory as in section 8 of [9] and we will choose a particular representative of the isomorphism class of \widehat{E} whose logarithm has a certain form. More precisely, let $f(X)$ be a *good lift* of Frobenius attached to π as in section 4.1 and let

$$\ell(X) := \sum_{k=0}^{\infty} (-1)^k \frac{f^{(2k)}(X)}{p^k} \in \mathbf{Q}_p[[X]],$$

where $f^{(0)}(X) = X$ and if $n \geq 1$ is an integer we set $f^{(n)}(X) := f(f^{(n-1)}(X))$. By Honda theory, if we denote by $G(X, Y) := \ell^{-1}(\ell(X) + \ell(Y))$, then \widehat{E} and G are isomorphic formal groups over \mathbf{Z}_p and the logarithm of G is $\ell(X)$. For the rest of this section we will identify these two formal groups and will write \widehat{E} for G . We first have

Lemma 4.6. *The formal group \widehat{E} has no p -power torsion points in k_n for all $n \geq 0$.*

Proof. The proof is the same, modulo the obvious adjustments, as the proof of Proposition 8.7 of [9]. \square

Corollary 4.7. *The group homomorphism $\ell : \widehat{E}(k_n) \longrightarrow \widehat{\mathbb{G}}_a(k_n)$ is injective.*

Proof. This corollary follows immediately from Lemma 4.6 since the kernel of the logarithm of a formal group is composed precisely of the elements of finite order. \square

Let $F_f(X, Y)$ be the Lubin-Tate formal group over \mathbf{Z}_p attached to the lift of Frobenius $f(X)$ as in section 4.1 and let us choose a π -sequence $\{e_n\}_{n \geq 0}$ in k_∞ , i.e. $e_n \in F_f[\pi^n] - F_f[\pi^{n-1}]$ such that $f(e_n) = e_{n-1}$ for all $n \geq 1$. Let $\epsilon \in p\mathbf{Z}_p$ be such that $\ell(\epsilon) = \frac{p}{p+1}$ and define $c_n \in \widehat{E}(k_n)$ to be $c_n = e_n[+]_{\widehat{E}}\epsilon$ for all $n \geq 0$. The following lemma computes the traces of the c_n .

Lemma 4.8. *For $n \geq 1$, we have $\text{Tr}_{n-1}^n(c_n) = -c_{n-2}$ where here Tr_{n-1}^n is the trace from $\widehat{E}(k_n)$ to $\widehat{E}(k_{n-1})$. For $n = 1$, $\text{Tr}_0^1(c_1) = u \cdot c_0$ with $u \in \mathbf{Z}_p^\times$.*

Proof. Everything is set up so that the proof follows formally the same steps as the proof of Lemma 8.9 in [9]. Namely, as ℓ is injective on $\widehat{E}(k_\infty)$, it is enough to show that the relation holds after applying ℓ to both sides of the equality. For $n \geq 2$, we have

$$\begin{aligned} \ell(\text{Tr}_{n-1}^n(c_n)) &= \text{Tr}_{k_n/k_{n-1}} \left(\frac{p}{p+1} + \sum_{k=0}^{\infty} (-1)^k \frac{e_{n-2k}}{p^k} \right) \\ &= \frac{p^2}{p+1} - p + p \sum_{k=1}^{\infty} (-1)^k \frac{e_{n-2k}}{p^k} \\ &= -\ell(c_{n-2}) \end{aligned}$$

where $e_k = 0$ for k negative. (Here we have used the fact that $\text{Tr}_{k_n/k_{n-1}}(e_n) = -p$ which follows from the fact that $f(x)$ is a *good lift* in the sense of section 4.1.) The calculation is similar for $n = 1$. \square

Proposition 4.9. *We have $\ell(M_n) \subset M_n + k_{n-1}$ and ℓ induces an isomorphism*

$$\widehat{E}(k_n)/\widehat{E}(k_{n-1}) \cong \ell(M_n)/\ell(M_{n-1}) \cong M_n/M_{n-1}.$$

Further c_n generates $\widehat{E}(k_n)/\widehat{E}(k_{n-1})$ as a $\mathbf{Z}_p[\text{Gal}(k_n/\mathbf{Q}_p)]$ -module.

Proof. The proof follows the steps of the proof of Proposition 8.11 of [9]. The main new ingredient is Proposition 4.4. \square

Corollary 4.10. *For $n \geq 1$, c_n and c_{n-1} generate $\widehat{E}(k_n)$ as a $\mathbf{Z}_p[\text{Gal}(k_n/\mathbf{Q}_p)]$ -module.*

Proof. This follows easily from Proposition 4.9 and the trace relations satisfied by the c_n (see Lemma 4.8). \square

Proof of Theorem 4.5. Let $d_n := \text{Tr}_{k_{n+1}/L_n}(c_{n+1}) \in \widehat{E}(L_n)$. Then it is easy to see that $\text{Tr}_{n-1}^n(d_n) = -d_{n-2}$ for $n \geq 2$. Moreover, since $[k_{n+1} : L_m]$ is prime to p , $\text{Tr}_{k_{n+1}/L_n} : \widehat{E}(k_{n+1}) \rightarrow \widehat{E}(L_n)$ is surjective. Thus, since c_n and c_{n-1} generate $\widehat{E}(k_{n+1})$ as a $\mathbf{Z}_p[\text{Gal}(k_{n+1}/\mathbf{Q}_p)]$ -module, we have that d_n and d_{n-1} generate $\widehat{E}(L_n)$ as a $\mathbf{Z}_p[\text{Gal}(L_n/\mathbf{Q}_p)]$ -module for $n \geq 1$. \square

The following proposition describes the relations that the d_n satisfy. We first introduce some notation that will be used throughout the remainder of the paper. Let $\Phi_n(X) := \sum_{i=0}^{p-1} X^{ip^{n-1}}$ be the p^n -th cyclotomic polynomial, $\xi_n = \Phi_n(1+X)$ and $\omega_n(X) := (X+1)^{p^n} - 1$. Also set,

$$\tilde{\omega}_n^+ := \prod_{\substack{1 \leq m \leq n \\ m \text{ even}}} \Phi_m(1+X), \quad \tilde{\omega}_n^- := \prod_{\substack{1 \leq m \leq n \\ m \text{ odd}}} \Phi_m(1+X),$$

$\omega_n^+ = X \cdot \tilde{\omega}_n^+$ and $\omega_n^- = X \cdot \tilde{\omega}_n^-$. Note that $\omega_n = X \cdot \tilde{\omega}_n^+ \cdot \tilde{\omega}_n^-$. Finally, set $\Lambda = \mathbf{Z}_p[[\text{Gal}(L_\infty/L)]]$ which we identify with $\mathbf{Z}_p[[X]]$ by choosing some topological generator of $\text{Gal}(L_\infty/L)$. Then $\Lambda_n := \Lambda/\omega_n\Lambda$ is identified with $\mathbf{Z}_p[\text{Gal}(L_n/L)]$.

Proposition 4.11. *There is an exact sequence*

$$0 \longrightarrow \widehat{E}(\mathbf{Q}_p) \longrightarrow \Lambda_n d_n \oplus \Lambda_{n-1} d_{n-1} \longrightarrow \widehat{E}(L_n) \longrightarrow 0$$

where the first map is the diagonal embedding (note that $\widehat{E}(\mathbf{Q}_p) \subseteq \Lambda_k d_k$ for each k) and the second map is $(a, b) \mapsto a - b$. Furthermore, $\Lambda_n d_n \cong \Lambda/\omega_n^\varepsilon$ with $\varepsilon = (-1)^n$.

Proof. The exact sequence comes from Proposition 8.12 of [9]. For the second part, we have that

$$\omega_n^\varepsilon d_n = \omega_{n-2}^\varepsilon (\xi_n \cdot d_n) = \omega_{n-2}^\varepsilon \text{Tr}_{n-1}^n(d_n) = -\omega_{n-2}^\varepsilon d_{n-2} = \cdots = \pm X d_0 = 0.$$

Hence, there is a surjective map $\Lambda_n/\omega_n^\varepsilon \rightarrow \Lambda_n d_n$ obtained by sending 1 to d_n . To see that this map is injective, it is enough to note that $\Lambda_n/\omega_n^\varepsilon$ and $\Lambda_n d_n$ are free \mathbf{Z}_p -modules of the same rank (which follows from the above exact sequence). \square

Corollary 4.12. *For $n \geq 0$ and $\varepsilon = (-1)^n$,*

1. $\ker(\text{Tr}_{n-1}^n) \cong \omega_{n-2}^\varepsilon \Lambda_n d_n$.
2. $\text{coker}(\text{Tr}_{n-1}^n)$ is a p -group with p -rank equal to q_n where

$$q_n = \begin{cases} p^{n-1} - p^{n-2} + \cdots + p - 1 & 2|n \\ p^{n-1} - p^{n-2} + \cdots + p^2 - p & 2 \nmid n \end{cases}.$$

Proof. By Proposition 4.11, we have

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widehat{E}(\mathbf{Q}_p) & \longrightarrow & \Lambda_n d_n \oplus \Lambda_{n-1} d_{n-1} & \longrightarrow & \widehat{E}(L_n) \longrightarrow 0 \\
& & \downarrow \times p & & \downarrow & & \downarrow \text{Tr}_{n-1}^n \\
0 & \longrightarrow & \widehat{E}(\mathbf{Q}_p) & \longrightarrow & \Lambda_{n-1} d_{n-1} \oplus \Lambda_{n-2} d_{n-2} & \longrightarrow & \widehat{E}(L_{n-1}) \longrightarrow 0
\end{array}$$

where the middle vertical map sends $(d_n, 0)$ to $(0, -d_{n-2})$ and $(0, d_{n-1})$ to $(p \cdot d_{n-1}, 0)$. Then applying the snake lemma and Proposition 4.11 yields the result. \square

4.3 The plus/minus Perrin-Riou map

We follow closely section 8 of [9] except that we work with a \mathbf{Z}_p -extension instead of a \mathbf{Z}_p^\times -extension. This produces a certain shift in the numbering but the main arguments are formally the same. Let T be the p -adic Tate-module of E considered as a $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ -module. The Kummer map $\widehat{E}(L_n) \longrightarrow H^1(L_n, T)$ together with cup product and the Weil pairing induces

$$(\ , \)_n : \widehat{E}(L_n) \times H^1(L_n, T) \longrightarrow H^2(L_n, \mathbf{Z}_p(1)) \cong \mathbf{Z}_p.$$

Let $G_n := \text{Gal}(L_n/\mathbf{Q}_p) \cong \mathbf{Z}/p^n\mathbf{Z}$ and for every $x \in \widehat{E}(L_n)$ let us define the morphism $P_{x,n} : H^1(L_n, T) \longrightarrow \mathbf{Z}_p[G_n]$ by $P_{x,n}(z) = \sum_{\sigma \in G_n} (x^\sigma, z)_n \sigma$. Both $H^1(L_n, T)$ and $\mathbf{Z}_p[G_n]$ are naturally G_n -modules and $P_{x,n}$ is G_n -equivariant for all x and n . Moreover, for every $x \in \widehat{E}(L_n)$ and $n \geq 1$ the following diagram

$$\begin{array}{ccc}
H^1(L_n, T) & \xrightarrow{P_{x,n}} & \mathbf{Z}_p[G_n] \\
\downarrow & & \downarrow \\
H^1(L_{n-1}, T) & \xrightarrow{P_{\text{Tr}_{n-1}^n(x), n-1}} & \mathbf{Z}_p[G_{n-1}]
\end{array}$$

is commutative. Using the sequence of points $\{d_n\}_n$ we consider two subsequences:

$$d_n^+ = \begin{cases} d_n & \text{if } n \text{ is even} \\ d_{n-1} & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad d_n^- = \begin{cases} d_{n-1} & \text{if } n \text{ is even} \\ d_n & \text{if } n \text{ is odd} \end{cases}$$

We set $P_n^\pm := (-1)^{\lfloor \frac{n+1}{2} \rfloor} P_{d_n^\pm, n}$ and define

$$\widehat{E}^+(L_n) := \{P \in \widehat{E}(L_n) \mid \text{Tr}_m^n(P) \in \widehat{E}(L_{m-1}) \text{ for all } 1 \leq m \leq n, m \text{ odd}\};$$

$$\widehat{E}^-(L_n) := \{P \in \widehat{E}(L_n) \mid \text{Tr}_m^n(P) \in \widehat{E}(L_{m-1}) \text{ for all } 1 \leq m \leq n, m \text{ even}\}.$$

Lemma 4.13. d_n^\pm generates $\widehat{E}^\pm(L_n)$ as a $\mathbf{Z}_p[G_n]$ -module.

Proof. The proof is the same as the proof of Proposition 8.12 of [9]. \square

We define $H_{\pm}^1(L_n, T) := \left(\widehat{E}(L_n)^{\pm} \otimes \mathbf{Q}_p/\mathbf{Z}_p \right)^{\perp} \subset H^1(L_n, T)$ where we think of $\widehat{E}(L_n)^{\pm} \otimes \mathbf{Q}_p/\mathbf{Z}_p$ as embedded in $H^1(L_n, V/T)$ by the Kummer map with $V = T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. The orthogonal complement is taken with respect to the Tate pairing $\langle \cdot, \cdot \rangle : H^1(L_n, T) \times H^1(L_n, V/T) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p$.

Lemma 4.14.

1. $\ker(P_n^{\pm}) = H_{\pm}^1(L_n, T)$.
2. The image of P_n^{\pm} is contained in $\tilde{\omega}_n^{\mp} \Lambda_n$.

Proof. The first part is clear from Lemma 4.13. For the second part, we have that

$$\omega_n^{\pm} P_n^{\pm}(z) = \omega_n^{\pm} \sum_{\sigma \in G_n} \left((d_n^{\pm})^{\sigma}, z \right)_n \sigma = \sum_{\sigma \in G_n} \left(\omega_n^{\pm} (d_n^{\pm})^{\sigma}, z \right)_n \sigma = 0$$

by Proposition 4.11. The lemma then follows because any element of Λ_n that is killed by ω_n^{\pm} is divisible by $\tilde{\omega}_n^{\mp}$. \square

Since $\omega_n = X \tilde{\omega}_n^+ \tilde{\omega}_n^-$, we have an isomorphism

$$\Lambda_n^{\pm} := \mathbf{Z}_p[X]/\omega_n^{\pm} \cong \tilde{\omega}_n^{\mp} \Lambda_n.$$

We define $P_{\Lambda_n}^{\pm}$ to be the unique map which makes the following diagram commute.

$$\begin{array}{ccc} H^1(L_n, T) & \xrightarrow{P_{\Lambda_n}^{\pm}} & \Lambda_n^{\pm} \\ \downarrow & & \downarrow \\ \frac{H^1(L_n, T)}{H_{\pm}^1(L_n, T)} & \xrightarrow{P_n^{\pm}} & \Lambda_n \end{array}$$

Here the right vertical map is $\Lambda_n^{\pm} \cong \tilde{\omega}_n^{\mp} \Lambda_n \subseteq \Lambda_n$. The properties of the maps $P_{\Lambda_n}^{\pm}$ are gathered in the following proposition.

Proposition 4.15.

1. For $n \geq 1$,

$$\begin{array}{ccc} H^1(L_{n+1}, T) & \xrightarrow{P_{\Lambda_{n+1}}^{\pm}} & \Lambda_{n+1}^{\pm} \\ \text{cor}_{n+1/n} \downarrow & & \downarrow \\ H^1(L_n, T) & \xrightarrow{P_{\Lambda_n}^{\pm}} & \Lambda_n^{\pm} \end{array} \quad (9)$$

commutes. (Here the right vertical map is the natural projection.)

2. $P_{\Lambda_n}^{\pm}$ is surjective for all $n \geq 1$.

3. $P_{\Lambda,n}^{\pm}$ determines an isomorphism $\frac{H^1(L_n, T)}{H_{\pm}^1(L_n, T)} \cong \Lambda_n^{\pm}$.

Proof. See the proofs of Proposition 8.18, 8.21 and 8.23 of [9]. □

Diagram (9) allows us to consider the projective limit (with respect to n) of the maps $P_{\Lambda,n}^{\pm}$ and we denote this limit by

$$P_{\Lambda}^{\pm} : \mathbf{H}^1(T) := \varprojlim_n (H^1(L_n, T), \text{cor}) \longrightarrow \varprojlim_n \Lambda_n^{\pm} \cong \Lambda.$$

Also, let $\mathbf{H}_{\pm}^1(T) := \varprojlim_n (H_{\pm}^1(L_n, T), \text{cor})$ and we have:

Proposition 4.16. P_{Λ}^{\pm} defines an isomorphism $\mathbf{H}^1(T)/\mathbf{H}_{\pm}^1(T) \cong \Lambda$. Furthermore, $\mathbf{H}_{\pm}^1(T)$ is a free Λ -module of rank 1.

Proof. For the first part, apply part (3) of Proposition 4.15. Then, from the first part, we know that $\mathbf{H}_{\pm}^1(T)$ is a direct summand of $\mathbf{H}^1(T)$ which by [16, Proposition 3.2.1] is a free Λ -module of rank 2. Therefore, $\mathbf{H}_{\pm}^1(T)$ is a projective Λ -module and since Λ is local, $\mathbf{H}_{\pm}^1(T)$ is free of rank 1. □

Finally, we have the following description of the maps P_n^{\pm} in terms of the dual exponential map of the Galois module T . (See section 8.7 of [9] for a discussion of the dual exponential map in this context.)

Proposition 4.17. *We have*

$$P_n^{\pm}(z) = \left(\sum_{\sigma \in G_n} \ell(d_n^{\pm})^{\sigma} \sigma \right) \left(\sum_{\sigma \in G_n} \exp_{\omega_E}^*(z^{\sigma}) \sigma^{-1} \right).$$

Proof. See Proposition 8.25 of [9]. □

5 The “most basic” case in Iwasawa theory

5.1 Algebraic results

In this section, we will be working under the following restrictive global hypothesis. Recall that p is assumed to be odd.

• **Hypothesis G** (for global):

1. $p \nmid \text{Tam}(E/K)$
2. $\text{III}(E/K)[p] = 0$
3. $E(K)/pE(K) = 0$.

In the good (non-anomalous) ordinary case, this hypothesis implies that both the μ -invariant and λ -invariant of E vanishes along any \mathbf{Z}_p -extension of K . For this reason, we refer to the situation in this section as the “most basic” case. Throughout this section we will be assuming (S) and (G) and under these hypotheses we will prove the following theorem.

Theorem 5.1. *Assuming (SG), $a_p = 0$ and p odd, we have*

1. $E(K_n)$ is finite
2. $(\text{III}(E/K_n)[p^\infty])^\wedge \cong (\Lambda/(\tilde{\omega}_n^+, \tilde{\omega}_n^-))^d$
3. $\text{ord}_p(\#\text{III}(E/K_n)[p^\infty]) = d \cdot \sum_{k=0}^n q_k$

where $d = [K : \mathbf{Q}]$ and q_k is defined in Corollary 4.12.

Remark 5.2. The hypothesis $a_p = 0$ is probably not necessary. See [19] for a proof of this theorem for general a_p (divisible by p) when $K = \mathbf{Q}$. However, the condition that p is odd is necessary (see [19, Remark 1.2]).

We begin by computing the structure of $\mathfrak{X}_\infty = (S_\infty)^\wedge = \text{Sel}(E[p^\infty]/K_\infty)^\wedge$ as a Λ -module. The following well known result does not assume (SG).

Proposition 5.3. *When p is supersingular for E/\mathbf{Q} ,*

$$\text{rk}_\Lambda \mathfrak{X}_\infty = \text{corank}_\Lambda H^1(K_\Sigma/K_\infty, E[p^\infty]) \geq d.$$

Proof. The first equality follows from [21, Corollary 5]. The inequality follows from a global Euler characteristic calculation (see [5, Proposition 3]) since

$$\text{corank}_\Lambda H^1(K_\Sigma/K_\infty, E[p^\infty]) - \text{corank}_\Lambda H^2(K_\Sigma/K_\infty, E[p^\infty]) = d.$$

□

Proposition 5.4. *Assuming (SG), \mathfrak{X}_∞ is a free Λ -module of rank d .*

Proof. Considering Theorem 3.1 with $n = 0$ yields

$$\widehat{E}(K_p) \twoheadrightarrow (\mathfrak{X}_\infty)_\Gamma \tag{10}$$

since, by (G), $\text{Sel}(E[p^\infty]/K) = 0$ and $p \nmid \text{Tam}(E/K)$. Now, $\text{rk}_{\mathbf{Z}_p}(\mathfrak{X}_\infty)_\Gamma \geq d$ by Proposition 5.3 and hence (10) is an isomorphism since $\widehat{E}(K_p) \cong \mathbf{Z}_p^d$. By Nakayama's lemma, we can lift (10) to a map $\Lambda^d \twoheadrightarrow \mathfrak{X}_\infty$. Again, by Proposition 5.3, $\text{rk}_\Lambda \mathfrak{X}_\infty \geq d$ and hence this map is an isomorphism. □

Remark 5.5. For the remainder of this section we will fix an isomorphism of \mathfrak{X}_∞ with Λ^d . Such an isomorphism (as constructed in Proposition 5.4) depends in part upon an identification of $\widehat{E}(K_p)$ with \mathbf{Z}_p^d . We will now specify this identification. By (S), $K_{\mathfrak{p}_j} \cong \mathbf{Q}_p$ and hence Theorem 4.5 applies to $\widehat{E}(K_{n,\mathfrak{p}_j})$. Set $d_{n,j} = (0, \dots, d_n, \dots, 0) \in \widehat{E}(K_{n,p}) = \bigoplus_{i=1}^d \widehat{E}(K_{n,\mathfrak{p}_i})$ where $d_n \in \widehat{E}(K_{n,\mathfrak{p}_j})$. Then $\{d_{0,j}\}_{j=1}^d$ generates $\widehat{E}(K_p)$ and in what follows we will assume that $\widehat{E}(K_p)$ is identified with \mathbf{Z}_p^d via these generators.

In particular, Theorem 3.1 yields

$$\widehat{E}(K_{n,p}) \xrightarrow{R_n} \Lambda_n^d \longrightarrow \mathfrak{X}_n \longrightarrow 0; \quad (11)$$

recall $\Lambda_n = \Lambda/\omega_n\Lambda$. Furthermore, for $m \leq n$ we have

$$\begin{array}{ccc} \widehat{E}(K_{n,p}) & \xrightarrow{R_n} & \Lambda_n^d \\ \text{Tr}_m^n \downarrow & & \downarrow \\ \widehat{E}(K_{m,p}) & \xrightarrow{R_m} & \Lambda_m^d \end{array} \quad (12)$$

where Tr_m^n is the trace map and the right vertical map is the natural projection. We postpone checking the commutativity of this diagram until section 6 (see Proposition 6.3).

Lemma 5.6. $\tilde{\omega}_n^{-\varepsilon} | R_n(d_{n,j})$ with $\varepsilon = (-1)^n$.

Proof. By Corollary 4.12, $d_{n,j}$ is killed by ω_n^ε . Since R_n is a Galois equivariant map, $R_n(d_{n,j})$ is also killed by ω_n^ε and is therefore divisible by $\tilde{\omega}_n^{-\varepsilon}$. \square

By Lemma 5.6, write

$$R_n(d_{n,j}) = \tilde{\omega}_n^{-\varepsilon} \cdot (u_{1j}, \dots, u_{dj}) \in \Lambda_n^d$$

where $\varepsilon = (-1)^n$.

Lemma 5.7. $\det(u_{ij})$ is a unit in Λ_n .

Proof. To prove this lemma it is enough to check that $\det(u_{ij}(0))$ is a unit in \mathbf{Z}_p . We have by diagram (12)

$$R_n(d_{n,j}) \equiv R_0(\text{Tr}_0^n(d_{n,j})) \text{ in } \Lambda_0^d \cong (\mathbf{Z}_p[X]/X)^d.$$

By Theorem 4.5, in the case that n is even, $\text{Tr}_0^n(d_{n,j}) = \pm p^{\frac{n}{2}} d_{0,j}$. Also by Remark 5.5, we have normalized R_0 so that $R_0(d_{0,j}) = (0, \dots, 1, \dots, 0)$ where 1 is in the j -th coordinate. Therefore, $R_n(d_{n,j})$ evaluated at 0 equals $(0, \dots, \pm p^{\frac{n}{2}}, \dots, 0)$.

On the other hand,

$$\begin{aligned} R_n(d_{n,j})(0) &= \tilde{\omega}_n^{-\varepsilon}(0) \cdot (u_{1j}(0), \dots, u_{dj}(0)) \\ &= p^{\frac{n}{2}} \cdot (u_{1j}(0), \dots, u_{dj}(0)). \end{aligned}$$

Therefore,

$$u_{ij}(0) = \begin{cases} 0 & i \neq j \\ \pm 1 & i = j \end{cases} \quad (13)$$

and $\det(u_{ij}(0)) = \pm 1 \in \mathbf{Z}_p^\times$. The case of n odd is proven similarly using the fact that $\text{Tr}_0^1(d_{1,j}) = u \cdot d_{0,j}$ with $u \in \mathbf{Z}_p^\times$. \square

Let $I_n = R_n(\widehat{E}(K_{n,p})) \subseteq \Lambda_n^d$. Then by Theorem 4.5, I_n is the ideal of Λ_n^d generated by $R_n(d_{n,j})$ and $R_n(d_{n-1,j})$ for $j = 1, \dots, d$.

Proposition 5.8. $\Lambda_n^d/I_n \cong (\Lambda/(\tilde{\omega}_n^+, \tilde{\omega}_n^-))^d$.

Proof. Let $\tilde{\omega}_{n,j}^\varepsilon = (0, \dots, \tilde{\omega}_n^\varepsilon, \dots, 0)$ where $\tilde{\omega}_n^\varepsilon$ lies in the j -th coordinate and let J_n be the ideal generated by $\tilde{\omega}_{n,j}^+$ and $\tilde{\omega}_{n,j}^-$ for $j = 1, \dots, d$. To prove the proposition, it suffices to show that $I_n = J_n$. By Proposition 5.6, $I_n \subseteq J_n$. Conversely, from (13) in the proof of Lemma 5.7, we have that $(I_n)_\Gamma \cong (J_n)_\Gamma$. Therefore, by Nakayama's lemma we can conclude $I_n = J_n$. \square

Proof of Theorem 5.1. From (11) and Proposition 5.8,

$$\mathfrak{X}_n \cong (\Lambda/(\tilde{\omega}_n^+, \tilde{\omega}_n^-))^d. \quad (14)$$

An explicit computation (see [10, Lemma 7.1]) shows that

$$\text{ord}_p \# (\Lambda/(\tilde{\omega}_n^+, \tilde{\omega}_n^-)) = \sum_{k=0}^n q_k. \quad (15)$$

Therefore, S_n is finite and, in particular, $E(K_n)$ is finite proving part (1). Now since there is no presence of rank, $S_n \cong \text{III}(E/K_n)[p^\infty]$; this together with (14) yields part (2). Finally, part (3) follows from (15). \square

5.2 Analytic consequences

We begin with a lemma that converts analytic hypotheses into algebraic ones. The following is a deep lemma that relies heavily upon Kato's Euler system.

Lemma 5.9. *If p is an odd supersingular prime for E/\mathbf{Q} such that*

1. $\text{ord}_p \left(\frac{L(E/\mathbf{Q}, 1)}{\Omega_{E/\mathbf{Q}}} \right) = 0$
2. $G_{\mathbf{Q}} \longrightarrow \text{Aut}(E[p])$ is surjective,

then $\text{Sel}(E[p^\infty]/\mathbf{Q}) = 0$ and $p \nmid \text{Tam}(E/\mathbf{Q})$.

Proof. We have that $L(E/\mathbf{Q}, 1) \neq 0$ and hence from Kato's Euler system [8], $E(\mathbf{Q})$ and $\text{III}(E/\mathbf{Q})$ are both finite. We must show that $\text{III}(E/\mathbf{Q})[p^\infty] = 0$ and $p \nmid \text{Tam}(E/\mathbf{Q})$.

The (analytic) p -adic L -function $L_p^{\text{an}}(E, T) \in \overline{\mathbf{Q}}_p[[T]]$ interpolates special values of L -series and in particular

$$L_p^{\text{an}}(E, 0) = \left(1 - \frac{1}{\alpha}\right)^2 \frac{L(E/\mathbf{Q}, 1)}{\Omega_{E/\mathbf{Q}}}$$

where α is a root of $x^2 - a_p x + p$ (see [13, Section 14]).

In [15], Perrin-Riou constructed an algebraic p -adic L -function $L_p^{\text{alg}}(E, T) \in \overline{\mathbf{Q}}_p[[T]]$ (defined up to a unit in Λ) with the property that

$$L_p^{\text{alg}}(E, 0) \sim \left(1 - \frac{1}{\alpha}\right)^2 \frac{\#\text{III}(E/\mathbf{Q}) \cdot \text{Tam}(E/\mathbf{Q})}{\#E^{\text{tor}}(\mathbf{Q})}$$

when $\text{Sel}(E[p^\infty]/\mathbf{Q})$ is finite (see also [17, Théorème 2.2.1]).

Kato proved a divisibility between these two p -adic L -functions under the above assumption on the Galois representation. Namely, we have that

$$L_p^{\text{alg}}(E, T) \mid L_p^{\text{an}}(E, T)$$

in $\mathbf{Z}_p[[T]]$ (see [8, Theorem 12.5] and [17, Théorème 3.1.3]). In particular,

$$\text{ord}_p(\text{III}(E/\mathbf{Q}) \cdot \text{Tam}(E/\mathbf{Q})) \leq \text{ord}_p\left(\frac{L(E/\mathbf{Q}, 1)}{\Omega_{E/\mathbf{Q}}}\right).$$

(Note that $E[p](\mathbf{Q}) = 0$ since p is supersingular.) From the above inequality, the lemma follows immediately since we are assuming that the right hand side is zero. \square

The following corollary, originally proven by Kurihara, follows from Theorem 5.1 and Lemma 5.9.

Corollary 5.10. *Let $K = \mathbf{Q}$ so that $K_\infty = \mathbf{Q}_\infty$ is the cyclotomic \mathbf{Z}_p -extension. Let E/\mathbf{Q} be an elliptic curve and p an odd prime of good reduction with $a_p = 0$. Assume that*

1. $\text{ord}_p\left(\frac{L(E/\mathbf{Q}, 1)}{\Omega_{E/\mathbf{Q}}}\right) = 0$
2. $G_{\mathbf{Q}} \longrightarrow \text{Aut}(E[p])$ is surjective.

Then the conclusions of Theorem 5.1 hold with $d = 1$.

Proof. First note that $\text{ord}_p\left(\frac{L(E/\mathbf{Q}, 1)}{\Omega_{E/\mathbf{Q}}}\right) = 0$ implies that $\text{Sel}(E[p^\infty]/\mathbf{Q}) = 0$ and that $p \nmid \text{Tam}(E/\mathbf{Q})$ by Lemma 5.9. Therefore, hypothesis (G) is satisfied. Furthermore, (S) is automatically satisfied when $K = \mathbf{Q}$ and the conclusions of Theorem 5.1 follow. \square

Corollary 5.11. *Let K be a quadratic extension of \mathbf{Q} and K_∞ any \mathbf{Z}_p -extension of K . Let E/\mathbf{Q} be an elliptic curve with p an odd prime of good reduction satisfying (S) for K and such that $a_p = 0$. Assume further that*

1. $\text{ord}_p\left(\frac{L(E/K, 1)}{\Omega_{E/K}}\right) = 0$
2. $G_K \longrightarrow \text{Aut}(E[p])$ is surjective.

Then the conclusions of Theorem 5.1 hold with $d = 2$.

Proof. Let E^D be the quadratic twist of E corresponding to K/\mathbf{Q} . Then $L(E/K, s) = L(E/\mathbf{Q}, s) \cdot L(E^D/\mathbf{Q}, s)$. In particular, $\text{ord}_p\left(\frac{L(E/K, 1)}{\Omega_{E/K}}\right) = 0$ implies that $\text{ord}_p\left(\frac{L(E/\mathbf{Q}, 1)}{\Omega_{E/\mathbf{Q}}}\right) = 0$ and $\text{ord}_p\left(\frac{L(E^D/\mathbf{Q}, 1)}{\Omega_{E^D/\mathbf{Q}}}\right) = 0$ since both special values are p -integral (see [18, Remark 6.5]). Since G_K surjects onto $\text{Aut}(E[p])$, we have that $G_{\mathbf{Q}}$ surjects onto both $\text{Aut}(E[p])$ and $\text{Aut}(E^D[p])$. Therefore, by Lemma 5.9, we have that $\text{Sel}(E[p^\infty]/\mathbf{Q}) = \text{Sel}(E^D[p^\infty]/\mathbf{Q}) = 0$ and that p does not divide $\text{Tam}(E/\mathbf{Q}) \cdot \text{Tam}(E^D/\mathbf{Q})$. From this we can conclude that $\text{Sel}(E[p^\infty]/K) = 0$ and that p does not divide $\text{Tam}(E/K)$. Therefore, hypothesis (G) is satisfied and the conclusions of Theorem 5.1 follow. \square

6 Algebraic p -adic L -functions

In this section, we construct algebraic p -adic L -functions in two different ways. First, we work directly with the points $\{d_n\}$ and Theorem 3.1 to produce two p -adic power series as in [14]. However, as in section 5, we first remove certain trivial zeroes to obtain elements of the Iwasawa algebra. Alternatively, we consider plus/minus Selmer groups as in [9] and define algebraic p -adic L -functions as the characteristic power series of these Λ -modules. Finally, we show that these two constructions yield the same power series (up to a unit in Λ). We continue to assume (S) in order to make use of the local results of section 4.

6.1 Construction of algebraic p -adic L -functions via $\{d_n\}$

We begin by generalizing the constructions done in section 5. Assuming (G), it was shown in Proposition 5.4 that $\text{rk}_\Lambda \mathfrak{X}_\infty = d$. In general, this would be true assuming a form of the weak Leopoldt conjecture. We introduce this conjecture as another hypothesis. (See [6] for a formulation of this conjecture and for cases when it is known to be true.)

- **Hypothesis W** (for Weak Leopoldt): $\text{corank}_\Lambda H^2(K_\Sigma/K_\infty, E[p^\infty]) = 0$.

Proposition 6.1. *When p is supersingular for E/\mathbf{Q} , we have that (W) is equivalent to $\text{rk}_\Lambda \mathfrak{X}_\infty = d$.*

Proof. This is clear from Proposition 5.3 and its proof. \square

If Y is the Λ -torsion submodule of \mathfrak{X}_∞ , we have

$$0 \longrightarrow Y \longrightarrow \mathfrak{X}_\infty \longrightarrow Z \longrightarrow 0 \quad (16)$$

where Z is torsion free. By Proposition 6.1, embedding Z into its reflexive hull yields a sequence

$$0 \longrightarrow Z \longrightarrow \Lambda^d \longrightarrow H \longrightarrow 0 \quad (17)$$

with H finite. We can then define a map

$$\widehat{E}(K_{n,p}) \longrightarrow \widehat{E}(K_{n,p}) \times B_n \longrightarrow (\mathfrak{X}_\infty)_{\Gamma_n} \longrightarrow Z_{\Gamma_n} \longrightarrow \Lambda_n^d$$

where the second map comes from Theorem 3.1, the third map comes from (16) and the final map comes from (17). Denote by Q_n the map from $\widehat{E}(K_{n,p})$ to $(\mathfrak{X}_\infty)_{\Gamma_n}$ and by R_n the map from $\widehat{E}(K_{n,p})$ to Λ_n^d . These maps satisfy an important compatibility property already exploited in section 5. Before discussing this property, we state a lemma on the functoriality of the snake lemma.

Lemma 6.2. *For $i = 1, 2$, let*

$$\begin{array}{ccccccc} A_i & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0 \\ a_i \downarrow & & b_i \downarrow & & c_i \downarrow & & \\ 0 & \longrightarrow & A'_i & \longrightarrow & B'_i & \longrightarrow & C'_i \end{array}$$

be a commutative diagram and assume that there are maps $A_1 \rightarrow A_2$, $B_1 \rightarrow B_2$, $C_1 \rightarrow C_2$ and likewise for A'_i , B'_i and C'_i such that all the respective squares commute. Then

$$\begin{array}{ccc} \ker(c_1) & \xrightarrow{\delta_1} & \text{coker}(a_1) \\ \downarrow & & \downarrow \\ \ker(c_2) & \xrightarrow{\delta_2} & \text{coker}(a_2) \end{array}$$

commutes where δ_i is the boundary map coming from the snake lemma.

Proof. This follows from a diagram chase. □

Proposition 6.3. *For $m \leq n$, we have that the following diagrams*

$$\begin{array}{ccc} \widehat{E}(K_{n,p}) & \xrightarrow{Q_n} & (\mathfrak{X}_\infty)_{\Gamma_n} & & \widehat{E}(K_{n,p}) & \xrightarrow{R_n} & \Lambda_n^d \\ \text{Tr}_m^n \downarrow & & \downarrow & & \text{Tr}_m^n \downarrow & & \downarrow \\ \widehat{E}(K_{m,p}) & \xrightarrow{Q_m} & (\mathfrak{X}_\infty)_{\Gamma_m} & & \widehat{E}(K_{m,p}) & \xrightarrow{R_m} & \Lambda_m^d \end{array}$$

commute.

Proof. Since we have a fixed map $\mathfrak{X}_\infty \rightarrow \Lambda^d$ defined independent of n , the following square

$$\begin{array}{ccc} (\mathfrak{X}_\infty)_{\Gamma_n} & \longrightarrow & \Lambda_n^d \\ \downarrow & & \downarrow \\ (\mathfrak{X}_\infty)_{\Gamma_m} & \longrightarrow & \Lambda_m^d \end{array}$$

commutes and therefore, we only need to check the commutativity of the left diagram in the proposition.

We will use the notation of Theorem 3.1. Furthermore, let $\widehat{E}_n = \widehat{E}(K_{n,p})$ and $M_n = \bigoplus_v \ker(r_{n,v})$. Then, examining the definition of $(Q_n)^\wedge$ piece-by-piece yields

$$\begin{array}{ccccccc} S_\infty^{\Gamma_m} & \twoheadrightarrow & \text{coker}(s_m) & \xrightarrow{\delta_m} & M_m \cap \text{im}(\gamma_m) & \subseteq & M_m \twoheadrightarrow \mathcal{H}_{m,p} \cong (\widehat{E}_m)^\wedge \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S_\infty^{\Gamma_n} & \twoheadrightarrow & \text{coker}(s_n) & \xrightarrow{\delta_n} & M_n \cap \text{im}(\gamma_n) & \subseteq & M_n \twoheadrightarrow \mathcal{H}_{n,p} \cong (\widehat{E}_n)^\wedge \end{array}$$

where the first horizontal map (for either the top or bottom row) is the natural projection, the second is given by the snake lemma (Proposition 3.2), the third is the natural inclusion, the fourth is the natural projection (applying Proposition 3.3) and the fifth is given by Tate local duality. The first vertical map is the natural inclusion, the second is induced by this inclusion, the third, fourth and fifth maps are induced by restriction and the sixth map is given by the dual of the trace map.

We now check the commutativity of this diagram square-by-square. The first square commutes essentially by definition. The second square commutes by the functoriality of the snake lemma (Lemma 6.2). The third and fourth squares commute because restriction commutes with these natural inclusions and projections. Finally, the commutativity of the last square is an essential property of Tate local duality (see [12, Proposition 4.2]). Dualizing then yields the proposition. \square

Since these maps are Galois equivariant, Proposition 5.6 remains valid in this setting. In particular, we can write

$$R_n(d_{n,j}) = \tilde{\omega}_n^{-\varepsilon} \cdot (-1)^{\lfloor \frac{n+1}{2} \rfloor} \cdot (u_{1j}^n, \dots, u_{dj}^n)$$

with $u_{ij}^n \in \Lambda/\omega_n^\varepsilon \Lambda$.

Lemma 6.4. *For $n > 1$ and $\varepsilon = (-1)^n$, $u_{ij}^n \equiv u_{ij}^{n-2} \pmod{\omega_{n-2}^\varepsilon}$.*

Proof. This lemma follows from Theorem 4.5 and Proposition 6.3. \square

From Lemma 6.4, we have that (u_{ij}^n) forms a compatible sequence inside of $\varprojlim_n \Lambda/\omega_n^{-\varepsilon} \Lambda$ for n running through positive integers of a fixed parity. When n is even, denote this sequence by u_{ij}^+ and when n is odd by u_{ij}^- . Since $\varprojlim_n \Lambda/\omega_n^\pm \Lambda \cong \Lambda$, we can consider u_{ij}^+ and u_{ij}^- as Iwasawa functions. We are now prepared to define the plus/minus algebraic p -adic L -functions.

Definition 6.5. Let Y be the Λ -torsion submodule of \mathfrak{X}_∞ and let $t_Y = \text{char}_\Lambda(Y)$. Then set

$$L_p^\pm(E, K_\infty/K, X) := \det(u_{ij}^\pm) \cdot t_Y$$

which is well-defined up to a unit in Λ .

Remark 6.6. Note that $L_p^\pm(E, K_\infty/K, X)$ can be identically zero. This vanishing occurs when $\text{corank}_{\mathbf{Z}_p}(S_n)$ is unbounded (see Corollary 7.10). Furthermore, these coranks can indeed be unbounded. For example, consider the case where K is a quadratic imaginary field and K_∞ is the anticyclotomic extension. The recent results of [2] and [25] show that if there are Heegner points present then indeed the corank of S_n will grow without bound.

6.2 Restricted Selmer groups

As in [9], we define plus/minus Selmer groups by putting harsher local conditions at each $\mathfrak{p}|p$.

Definition 6.7. Set

$$\mathrm{Sel}^\pm(E[p^\infty]/K_n) = \ker \left(\mathrm{Sel}(E[p^\infty]/K_n) \longrightarrow \prod_{\mathfrak{p}|p} \frac{E(K_{n,\mathfrak{p}}) \otimes \mathbf{Q}_p/\mathbf{Z}_p}{\widehat{E}^\pm(K_{n,\mathfrak{p}}) \otimes \mathbf{Q}_p/\mathbf{Z}_p} \right)$$

and $\mathrm{Sel}^\pm(E[p^\infty]/K_\infty) = \varinjlim_n \mathrm{Sel}^\pm(E[p^\infty]/K_n)$.

These plus/minus Selmer groups behave like Selmer groups at ordinary primes. In particular, they satisfy a control theorem in the spirit of Mazur's original control theorem.

Theorem 6.8. *The natural map*

$$\mathrm{Sel}^\pm(E[p^\infty]/K_n)^{\omega_n^\pm=0} \longrightarrow \mathrm{Sel}^\pm(E[p^\infty]/K_\infty)^{\omega_n^\pm=0}$$

is injective and has a finite cokernel bounded independent of n .

Proof. The proof in [9, Theorem 9.3] translates verbatim over to our situation. \square

If $\mathfrak{X}_\infty^\pm = \mathfrak{X}_\infty^\pm(E/K_\infty) = \mathrm{Sel}^\pm(E[p^\infty]/K_\infty)^\wedge$, then \mathfrak{X}_∞^\pm need not be a torsion Λ -module. The ranks of these modules will be discussed in section 7.1.

6.3 Comparing $\mathrm{Sel}^\pm(E[p^\infty]/K_\infty)$ and $L_p^\pm(E, K_\infty/K, X)$

As in the ordinary case, when \mathfrak{X}_∞^\pm is a torsion module, its characteristic power series should be considered as an algebraic p -adic L -function. The following proposition (whose proof will fill the remainder of the section) relates this point of view with that of section 6.1.

Proposition 6.9. *Assuming (W),*

$$\mathrm{char}_\Lambda \mathfrak{X}_\infty^\pm = L_p^\pm(E, K_\infty/K, X) \cdot v.$$

with $v \in \Lambda^\times$.

Before proving this proposition, we begin with a few lemmas.

Lemma 6.10. *For $\mathfrak{p}|p$ and $n \geq 0$,*

$$\tilde{\omega}_n^{-\varepsilon} \cdot H_\varepsilon^1(K_{n,\mathfrak{p}}, T) \subseteq \widehat{E}^\varepsilon(K_{n,\mathfrak{p}})$$

with $\varepsilon = (-1)^n$.

Proof. We check this for n even; the case of n odd is similar. We have

$$\frac{H_{\pm}^1(K_{n,p}, T)}{\widehat{E}(K_{n,p})} \hookrightarrow \frac{H^1(K_{n,p}, T)}{H_{\mp}^1(K_{n,p}, T)} \cong \Lambda/\omega_n^{\mp} \Lambda$$

where the second map is given by Proposition 4.15. The first map is injective since $z \in H_{+}^1(K_{n,p}, T) \cap H_{-}^1(K_{n,p}, T)$ is orthogonal to $\widehat{E}(K_{n,p}) = \widehat{E}^{+}(K_{n,p}) + \widehat{E}^{-}(K_{n,p})$ and hence in $\widehat{E}(K_{n,p})$. However, this map is not surjective; its image is killed by $\tilde{\omega}_n^{-}$ (rather than just ω_n^{-}) which we now check.

Note that $\frac{H^1(K_{n,p}, T)}{H_{\mp}^1(K_{n,p}, T)} \cong \Lambda/\omega_n^{\mp} \Lambda$ is free over \mathbf{Z}_p and hence $\frac{H_{\pm}^1(K_{n,p}, T)}{\widehat{E}(K_{n,p})}$, being a submodule, is also free. Then since $\widehat{E}(K_{n,p})$ is free, we can conclude that $H_{\pm}^1(K_{n,p}, T)$ is free. Now

$$\mathrm{rk}_{\mathbf{Z}_p} \frac{H^1(K_{n,p}, T)}{H_{+}^1(K_{n,p}, T)} = \mathrm{rk}_{\mathbf{Z}_p} \Lambda/\omega_n^{+} \Lambda = \deg(\omega_n^{+}) = p^n - p^{n-1} + \cdots + p^2 - p + 1.$$

Hence, since $\mathrm{rk}_{\mathbf{Z}_p} H^1(K_{n,p}, T) = 2 \cdot p^n$,

$$\mathrm{rk}_{\mathbf{Z}_p} H_{+}^1(K_{n,p}, T) = 2 \cdot p^n - (p^n - p^{n-1} + \cdots + p^2 - p + 1) = p^n + q_n$$

and therefore $\frac{H_{+}^1(K_{n,p}, T)}{\widehat{E}(K_{n,p})}$ has \mathbf{Z}_p -rank equal to q_n .

Now, if M is any submodule of

$$\Lambda/\omega_n^{-} \Lambda \cong \mathbf{Z}_p[X]/\omega_n^{-}(X) \cong (\oplus_{k \text{ odd}} \mathbf{Z}_p[\mu_{p^k}]) \oplus \mathbf{Z}_p,$$

then the projection of M to $\mathbf{Z}_p[\mu_{p^k}]$ for k odd or $k = 0$ is an ideal of this ring. In particular, this projection is zero or of finite index.

If M equals the image of $\frac{H_{+}^1(K_{n,p}, T)}{\widehat{E}(K_{n,p})}$ in $\Lambda/\omega_n^{-} \Lambda$, then since the \mathbf{Z}_p -rank of M is equal to q_n , its projection to $\mathbf{Z}_p[\mu_{p^k}]$ must be non-zero for $k > 0$ and its projection to \mathbf{Z}_p must be zero. But this means precisely that $\tilde{\omega}_n^{-}$ annihilates M and thus

$$\tilde{\omega}_n^{-} \cdot H_{+}^1(K_{n,p}, T) \subseteq \widehat{E}(K_{n,p}).$$

Since $\omega_n^{+} \cdot (\tilde{\omega}_n^{-} \cdot H_{+}^1(K_{n,p}, T)) = \omega_n \cdot H_{+}^1(K_{n,p}, T) = 0$, we further have that $\tilde{\omega}_n^{-} \cdot H_{+}^1(K_{n,p}, T) \subseteq \widehat{E}^{+}(K_{n,p})$ by the definition of \widehat{E}^{+} ; this completes the proof. \square

Repeating the arguments of Theorem 3.1 for the plus/minus Selmer groups yields

$$\begin{array}{ccccccc} B_n \times \oplus_{\mathfrak{p}|p} H_{\pm}^1(K_{n,p}, T) & \xrightarrow{Q_n^{\pm}} & (\mathfrak{X}_{\infty})_{\Gamma_n} & \longrightarrow & \mathfrak{X}_n^{\pm} & \longrightarrow & 0 \\ & \uparrow & \uparrow = & & \uparrow & & \\ B_n \times \widehat{E}(K_{n,p}) & \xrightarrow{Q_n} & (\mathfrak{X}_{\infty})_{\Gamma_n} & \longrightarrow & \mathfrak{X}_n & \longrightarrow & 0 \end{array} \quad (18)$$

where $\mathfrak{X}_n^{\pm} = \mathrm{Sel}^{\pm}(E[p^{\infty}]/K_n)^{\wedge}$. Taking the projective limit of the top line of the above diagram yields

$$\oplus_{\mathfrak{p}|p} \mathbf{H}_{\pm}^1(K_{\infty,p}, T) \xrightarrow{Q^{\pm}} \mathfrak{X}_{\infty} \longrightarrow \mathfrak{X}_{\infty}^{\pm}(E/K_{\infty}) \longrightarrow 0. \quad (19)$$

Let R^\pm be the composition of Q^\pm with the embedding of \mathfrak{X}_∞ into Λ^d from (16) and (17) and define R_n^\pm similarly. By Proposition 4.16, $\mathbf{H}_\pm^1(K_{\infty,p}, T)$ is a free Λ -module of rank 1.

Lemma 6.11. *For each j , fix a generator z_j of $\mathbf{H}_\pm^1(K_{\infty,p_j}, T)$. Then*

$$R^\pm(z_j) = (u_{1j}^\pm, \dots, u_{dj}^\pm) \cdot v_j^\pm$$

with v_j^\pm a unit in Λ .

Proof. Let $\varepsilon = (-1)^n$. We begin by recovering the sequence $\{d_{n,j}\}_n$ (constructed in section 4) from the element z_j . Let z_j^n be the image of z_j in $H_\pm^1(K_{n,p_j}, T)$ and let $d'_{n,j} = (-1)^{\lfloor \frac{n+1}{2} \rfloor} \cdot \tilde{\omega}_n^\varepsilon \cdot z_j^n$.

Claim: $d'_{n,j} = d_{n,j} \cdot v_j^\varepsilon$ for v_j^ε a unit in Λ (depending only on the parity of n).

First note that by Lemma 6.10, $d'_{n,j}$ is in fact an element of $\widehat{E}^\varepsilon(K_{n,p_j})$. Furthermore, the z_j^n are compatible under corestriction by construction. Therefore,

$$\mathrm{Tr}_{n-2}^n(d'_{n,j}) = \mathrm{Tr}_{n-2}^n(\tilde{\omega}_n^\varepsilon z_j^n) = p \cdot \tilde{\omega}_{n-2}^\varepsilon z_j^{n-2} = -p \cdot d'_{n-2,j}. \quad (20)$$

Since $\widehat{E}^\varepsilon(K_{n,p})$ is cyclic, generated by $d_{n,j}$ (Lemma 4.13), we can write $d'_{n,j} = d_{n,j} \cdot v_{n,j}$ with $v_{n,j} \in \Lambda/\omega_n^\varepsilon \Lambda$. Then (20) implies that $(v_{n,j})_n$ forms a compatible sequence for n of a fixed parity. Call the limiting function in $\varprojlim_n \Lambda/\omega_n^\varepsilon \Lambda \cong \Lambda$ by v_j^+ for n even and by v_j^- for n odd. To establish the claim it remains to show that v_j^\pm is a unit.

By [10, Proposition 9.2], $\mathbf{H}_\pm^1(K_{\infty,p_j}, T)$ surjects onto $H_\pm^1(K_{p_j}, T) \cong \widehat{E}(K_{p_j})$. Therefore, z_j^0 (resp. $\mathrm{Tr}_0^1(z_j^1)$) generates $\widehat{E}(K_{p_j})$. In particular, $d'_{0,j}$ (resp. $\mathrm{Tr}_0^1(d'_{1,j})$) differs from $d_{0,j}$ (resp. $\mathrm{Tr}_0^1(d_{1,j})$) by a unit in \mathbf{Z}_p . Hence, $v_j^\pm(0) \in \mathbf{Z}_p^\times$ and v_j^\pm is a unit in Λ .

By the claim,

$$\begin{aligned} \tilde{\omega}_n^\varepsilon \cdot R_n^\varepsilon(z_j^n) &= R_n \left((-1)^{\lfloor \frac{n+1}{2} \rfloor} d'_{n,j} \right) = R_n \left((-1)^{\lfloor \frac{n+1}{2} \rfloor} d_{n,j} \right) \cdot v_j^\varepsilon \\ &= \tilde{\omega}_n^\varepsilon \cdot (u_{1j}^n, \dots, u_{dj}^n) \cdot v_j^\varepsilon. \end{aligned}$$

Then cancelling $\tilde{\omega}_n^\varepsilon$ and taking limits over n of a fixed parity yields the lemma. \square

Proof of Proposition 6.9. If $\mathrm{corank}_{\mathbf{Z}_p}(S_n)$ is unbounded we will see by Corollary 7.7 and Corollary 7.10 that our proposition holds with $(0) = (0)$. So we may assume that $\mathrm{corank}_{\mathbf{Z}_p}(S_n)$ is bounded. From (19), we have

$$\begin{aligned} \mathrm{char}_\Lambda \mathfrak{X}_\infty^\pm &= \mathrm{char}_\Lambda \left(\frac{\mathfrak{X}_\infty}{Q^\pm (\oplus \mathbf{H}_\pm^1(K_{\infty,p_j}, T))} \right) \\ &= \mathrm{char}_\Lambda \left(\frac{\Lambda^d}{\{R^\pm(z_j)\}_{j=1}^d} \right) \cdot \mathrm{char}_\Lambda Y \end{aligned}$$

since $\mathfrak{X}_\infty/Y \subseteq \Lambda^d$ has finite index. Then by Lemma 6.11, $\text{char}_\Lambda \left(\frac{\Lambda^d}{\{R^\pm(z_j)\}_{j=1}^d} \right) = \det(u_{ij}^\pm) \cdot v$ with $v = \prod_j v_j^\pm \in \Lambda^\times$. Hence, $\text{char}_\Lambda \mathfrak{X}_\infty^\pm = \det(u_{ij}^\pm) \cdot \text{char}_\Lambda Y \cdot v = L_p^\pm(E, K_\infty/K, X) \cdot v$ which completes the proof. \square

7 Growth of Selmer groups in \mathbf{Z}_p -extensions

In this section, we explore the growth of $\text{corank}_{\mathbf{Z}_p}(S_n)$ as n varies. We describe this growth in terms of the Λ -ranks of \mathfrak{X}_∞^+ and \mathfrak{X}_∞^- . When $\text{corank}_{\mathbf{Z}_p}(S_n)$ is bounded, we compute the growth of $\text{III}(E/K_n)[p^\infty]$ in terms of the μ and λ -invariants of $L_p^\pm(E, K_\infty/K, X)$ as in [17]. Throughout this section, we will be assuming (S).

7.1 Corank of Selmer groups

Let $r^\pm = \text{rk}_\Lambda \mathfrak{X}_\infty^\pm = \text{corank}_\Lambda \text{Sel}^\pm(E[p^\infty]/K_\infty)$.

Proposition 7.1. *Assuming (W), we have*

$$\text{corank}_{\mathbf{Z}_p}(S_n) = r^\varepsilon \cdot q_{n+1} + r^{-\varepsilon} \cdot q_n + O(1)$$

where $\varepsilon = (-1)^n$. (Here, and in what follows, the $O(1)$ term depends upon E and upon K_∞/K , but not upon n .)

Remark 7.2. Note that if $r^+ = r^-$, then $\text{corank}_{\mathbf{Z}_p} S_n = r^\pm \cdot p^n + O(1)$ since $q_{n+1} + q_n = p^n - 1$. In the ordinary case, such growth formulas always have this form. However, in the supersingular case, one should have situations where $r^+ \neq r^-$. Namely, if K is a quadratic imaginary extension of \mathbf{Q} , K_∞ is the anticyclotomic \mathbf{Z}_p -extension and E has CM by K , then conjecturally $r^\varepsilon = 1$ and $r^{-\varepsilon} = 0$ where ε is minus the sign of the functional equation for E . (See [7, pg. 247] and [1].)

Before proving Proposition 7.1, we begin with a definition and some lemmas.

Definition 7.3. For L a finite extension of \mathbf{Q} , let

$$\begin{aligned} \text{Sel}^0(E[p^\infty]/L) &= \ker \left(\text{Sel}(E[p^\infty]/L) \longrightarrow \prod_{\mathfrak{p}|p} E(L_{\mathfrak{p}}) \otimes \mathbf{Q}_{\mathfrak{p}}/\mathbf{Z}_{\mathfrak{p}} \right); \\ \text{Sel}^1(E[p^\infty]/L) &= \ker \left(\text{Sel}(E[p^\infty]/L) \longrightarrow \prod_{\mathfrak{p}|p} \frac{E(L_{\mathfrak{p}}) \otimes \mathbf{Q}_{\mathfrak{p}}/\mathbf{Z}_{\mathfrak{p}}}{E(\mathbf{Q}_{\mathfrak{p}}) \otimes \mathbf{Q}_{\mathfrak{p}}/\mathbf{Z}_{\mathfrak{p}}} \right). \end{aligned}$$

To ease notation, let $S_n^\pm = \text{Sel}^\pm(E[p^\infty]/K_n)$, $S_n^0 = \text{Sel}^0(E[p^\infty]/K_n)$ and $S_n^1 = \text{Sel}^1(E[p^\infty]/K_n)$.

Lemma 7.4. *We have*

1. For any finitely generated Λ_n -module M , the map $M^{\omega_n^\varepsilon=0} \oplus M^{\tilde{\omega}_n^{-\varepsilon}=0} \longrightarrow M$ has finite kernel and cokernel.
2. $(S_n^\pm)^{\tilde{\omega}_n^\mp=0} \subseteq S_n^1$.
3. The map $S_n^+ \oplus S_n^- \longrightarrow S_n$ has finite cokernel and its kernel is contained in S_n^1 .

Proof. Part (1) follows from the fact that $\Lambda_n/(\omega_n^\varepsilon, \tilde{\omega}_n^{-\varepsilon})$ is finite. To see part (2), note that if $\sigma \in S_n^\pm$ is killed by $\tilde{\omega}_n^\mp$, then the restriction of σ to any \mathfrak{p} over p will lie in

$$(E^+(K_{n,\mathfrak{p}}) \otimes \mathbf{Q}_p/\mathbf{Z}_p) \cap (E^-(K_{n,\mathfrak{p}}) \otimes \mathbf{Q}_p/\mathbf{Z}_p) = E(K_{\mathfrak{p}}) \otimes \mathbf{Q}_p/\mathbf{Z}_p.$$

Part (3) follows as in [9, Proposition 10.1]. \square

Lemma 7.5. *Assuming (W), $\text{corank}_{\mathbf{Z}_p} S_n^1$ is bounded independent of n .*

Proof. Since $\text{coker}(S_n^0 \longrightarrow S_n^1)$ has \mathbf{Z}_p -rank bounded by d , it suffices to check that $\text{corank}_{\mathbf{Z}_p} S_n^0$ is bounded. By Theorem 3.1, we have that

$$\text{rk}_{\mathbf{Z}_p} S_n(T) - \text{rk}_{\mathbf{Z}_p} S_{n,\Sigma}(T) + \text{rk}_{\mathbf{Z}_p} (\mathfrak{X}_\infty)_{\Gamma_n} = \text{rk}_{\mathbf{Z}_p} \widehat{E}(K_{n,p}) + \text{rk}_{\mathbf{Z}_p} \mathfrak{X}_n.$$

By (W), $\text{rk}_\Lambda \mathfrak{X}_\infty = d$ and hence $\text{rk}_{\mathbf{Z}_p} (\mathfrak{X}_\infty)_{\Gamma_n} = d \cdot p^n + O(1)$. Furthermore, we have that

$$\text{rk}_{\mathbf{Z}_p} S_n(T) = \text{rk}_{\mathbf{Z}_p} \mathfrak{X}_n, \quad \text{rk}_{\mathbf{Z}_p} S_{n,\Sigma}(T) = \text{rk}_{\mathbf{Z}_p} S_n^0 \quad \text{and} \quad \text{rk}_{\mathbf{Z}_p} \widehat{E}(K_{n,p}) = d \cdot p^n.$$

Hence, $\text{corank}_{\mathbf{Z}_p} S_n^0$ is $O(1)$ (i.e. bounded). \square

Proof of Proposition 7.1. By part (3) of Lemma 7.4, we have

$$\text{corank}_{\mathbf{Z}_p} S_n \leq \text{corank}_{\mathbf{Z}_p} S_n^\varepsilon + \text{corank}_{\mathbf{Z}_p} S_n^{-\varepsilon} + \text{corank}_{\mathbf{Z}_p} S_n^1$$

and thus

$$\text{corank}_{\mathbf{Z}_p} S_n = \text{corank}_{\mathbf{Z}_p} S_n^\varepsilon + \text{corank}_{\mathbf{Z}_p} S_n^{-\varepsilon} + O(1)$$

by Lemma 7.5.

Applying part (1) and part (2) of Lemma 7.4 (and again Lemma 7.5) yields

$$\text{corank}_{\mathbf{Z}_p} S_n = \text{corank}_{\mathbf{Z}_p} (S_n^\varepsilon)^{\omega_n^\varepsilon=0} + \text{corank}_{\mathbf{Z}_p} (S_n^{-\varepsilon})^{\omega_n^{-\varepsilon}=0} + O(1). \quad (21)$$

Then, for $\varepsilon = (-1)^n$, by Theorem 6.8,

$$\begin{aligned} \text{corank}_{\mathbf{Z}_p} (S_n^\varepsilon)^{\omega_n^\varepsilon=0} &= \text{rk}_{\mathbf{Z}_p} (\mathfrak{X}_\infty^\varepsilon / \omega_n^\varepsilon \mathfrak{X}_\infty^\varepsilon) + O(1) \\ &= r^\varepsilon \cdot \text{rk}_{\mathbf{Z}_p} (\Lambda / \omega_n^\varepsilon \Lambda) + O(1) \\ &= r^\varepsilon \cdot q_{n+1} + O(1). \end{aligned}$$

Similarly,

$$\text{corank}_{\mathbf{Z}_p} (S_n^{-\varepsilon})^{\omega_n^{-\varepsilon}=0} = r^{-\varepsilon} \cdot q_n + O(1).$$

Substituting back into (21) then yields the proposition. \square

When $\text{corank}_{\mathbf{Z}_p} S_n$ is bounded, we will see that \mathfrak{X}_∞^+ and \mathfrak{X}_∞^- are Λ -torsion. We introduce this condition as another hypothesis.

- **Hypothesis B** (for bounded): $\text{corank}_{\mathbf{Z}_p}(S_n)$ is bounded.

Lemma 7.6. *(B) implies (W).*

Proof. Let $r = \text{rk}_\Lambda \mathfrak{X}_\infty$. By Proposition 5.3 it suffices to check that $r = d$. We have that $\text{rk}_{\mathbf{Z}_p} \widehat{E}(K_{n,p}) = dp^n$ and by the theory of Λ -modules, $\text{rk}_{\mathbf{Z}_p}(\mathfrak{X}_\infty)_{\Gamma_n}$ grows like rp^n . Then from Theorem 3.1 and (B), we can conclude $r = d$ completing the proof. \square

Corollary 7.7. *Hypothesis (B) holds if and only if \mathfrak{X}_∞^+ and \mathfrak{X}_∞^- are torsion Λ -module.*

Proof. If \mathfrak{X}_∞^+ or \mathfrak{X}_∞^- are not torsion, then by Theorem 6.8, (B) must fail. Conversely, if (B) holds, then (W) holds. Hence, by Proposition 7.1, $r^+ = r^- = 0$ and \mathfrak{X}_∞^\pm is torsion. \square

7.2 p -cyclotomic zeroes of $L_p^\pm(E, K_\infty/K, X)$

We now relate certain zeroes of the p -adic L -function $L_p^\pm(E, K_\infty/K, X)$ to the corank of S_n . Denote by ζ_n a primitive p^n -th root of unity and let $\xi_n = \Phi_n(1 + X)$.

Lemma 7.8. *Let $u_j = (u_{1j}, \dots, u_{dj}) \in \Lambda^d$ for $j = 1, \dots, d$. Then we have $(\Lambda/\xi_n)^d / (u_1, \dots, u_d)$ is finite if and only if $\det(u_{ij}(\zeta_n - 1)) \neq 0$. When this occurs*

$$\text{ord}_p \left(\# \frac{(\Lambda/\xi_n)^d}{(u_1, \dots, u_d)} \right) = \mu(f) \cdot (p^n - p^{n-1}) + \lambda(f)$$

where $f = \det(u_{ij})$ and n is sufficiently large.

Proof. Set $u_j(\zeta_n - 1) = (u_{1j}(\zeta_n - 1), \dots, u_{dj}(\zeta_n - 1))$ and then

$$\frac{(\Lambda/\xi_n)^d}{(u_1, \dots, u_d)} \cong \frac{\mathbf{Z}_p[\mu_{p^n}]^d}{(u_1(\zeta_n - 1), \dots, u_d(\zeta_n - 1))}.$$

By linear algebra, the left hand side is finite if and only if $\det(u_{ij}(\zeta_n - 1)) \neq 0$. Furthermore, when these groups are finite, they have size $(p^n - p^{n-1}) \cdot \text{ord}_p(\det(u_{ij}(\zeta_n - 1)))$ since p is totally ramified in $\mathbf{Z}_p[\mu_{p^n}]$. Our result then follows since for any non-zero $g \in \Lambda$,

$$\text{ord}_p(g(\zeta_n - 1)) = \mu(g) + \frac{\lambda(g)}{(p^n - p^{n-1})}$$

for n large enough. \square

Proposition 7.9. *We have that*

1. $L_p^+(E, K_\infty/K, 0) \neq 0$ and $L_p^-(E, K_\infty/K, 0) \neq 0$ if and only if S_0 is finite.
2. For $n > 1$, $L_p^\varepsilon(E, K_\infty/K, \zeta_n - 1) \neq 0$ for $\varepsilon = (-1)^n$ if and only if S_n/S_{n-1} is finite.

Proof. We prove this for even $n > 1$; the other cases follow similarly. Consider the diagram

$$\begin{array}{ccccccc}
\widehat{E}(K_{n,p}) & \xrightarrow{Q_n} & (\mathfrak{X}_\infty)_{\Gamma_n} & \longrightarrow & \mathfrak{X}_n & \longrightarrow & 0 \\
\text{Tr}_{n-1}^n \downarrow & & \downarrow \pi_n & & \downarrow & & \\
\widehat{E}(K_{n-1,p}) & \xrightarrow{Q_{n-1}} & (\mathfrak{X}_\infty)_{\Gamma_{n-1}} & \longrightarrow & \mathfrak{X}_{n-1} & \longrightarrow & 0
\end{array} \tag{22}$$

Then S_n/S_{n-1} is finite if and only if $\ker(\pi_n)/Q_n(\ker(\text{Tr}_{n-1}^n))$ is finite by Corollary 4.12. We have that $\ker(\pi_n) \cong \omega_{n-1}\mathfrak{X}_\infty/\omega_n\mathfrak{X}_\infty$ and from (16)

$$0 \longrightarrow \frac{\omega_{n-1}Y}{\omega_n Y} \longrightarrow \frac{\omega_{n-1}\mathfrak{X}_\infty}{\omega_n \mathfrak{X}_\infty} \longrightarrow \frac{\omega_{n-1}Z}{\omega_n Z} \longrightarrow 0. \tag{23}$$

The map R_n restricted to $\ker(\text{Tr}_{n-1}^n)$ is given by the composite map

$$\ker(\text{Tr}_{n-1}^n) \xrightarrow{Q_n} \frac{\omega_{n-1}\mathfrak{X}_\infty}{\omega_n \mathfrak{X}_\infty} \longrightarrow \frac{\omega_{n-1}Z}{\omega_n Z} \longrightarrow \left(\frac{\omega_{n-1}\Lambda}{\omega_n \Lambda} \right)^d. \tag{24}$$

Now by Corollary 4.12, $\{\omega_{n-2}^+ d_{n,j}\}_{j=1}^d$ generates $\ker(\text{Tr}_{n-1}^n)$ and we have that $R_n(\omega_{n-2}^+ d_{n,j}) = \omega_{n-1} \cdot (u_{1j}^n, \dots, u_{dj}^n)$. Set $u_j^n = (u_{1j}^n, \dots, u_{dj}^n) \in (\Lambda/\xi_n)^d$.

First we consider the case where $t_Y(\zeta_n - 1) = 0$. Then by definition $L_p^+(E, K_\infty/K, \zeta_n - 1) = 0$ and we need to check that S_n/S_{n-1} is infinite. Since $t_Y(\zeta_n - 1) = 0$, we have that $\omega_{n-1}Y/\omega_n Y$ is infinite. Then by Proposition 6.1 and (23)

$$\text{rk}_{\mathbf{Z}_p}(\omega_n \mathfrak{X}_\infty / \omega_{n-1} \mathfrak{X}_\infty) > d \cdot (p^n - p^{n-1}).$$

But $\text{rk}_{\mathbf{Z}_p}(\ker(\text{Tr}_{n-1}^n)) = d \cdot (p^n - p^{n-1})$ and hence S_n/S_{n-1} is infinite from (22).

So we may assume that $t_Y(\zeta_n - 1) \neq 0$. Then $L_p^+(E, K_\infty/K, \zeta_n - 1) \neq 0$ is equivalent to $\det(u_{ij}^n(\zeta_n - 1)) \neq 0$ which by Lemma 7.8 is equivalent to $(\Lambda/\xi_n)^d / (u_1^n, \dots, u_d^n)$ being finite. Since the last two maps in (24) have finite kernel and cokernel, these last statements are equivalent to

$$\frac{\omega_n \mathfrak{X}_\infty / \omega_{n-1} \mathfrak{X}_\infty}{(Q_n(d_{n,1}), \dots, Q_n(d_{n,d}))} = \frac{\ker(\pi_n)}{Q_n(\ker(\text{Tr}_{n-1}^n))} \text{ being finite.}$$

Then by (22), this is equivalent to S_n/S_{n-1} being finite completing the proof. \square

Corollary 7.10. $L_p^+(E, K_\infty/K, X) \neq 0$ and $L_p^-(E, K_\infty/K, X) \neq 0$ if and only if $\text{corank}_{\mathbf{Z}_p}(S_n)$ is bounded.

Proof. The result follows from Proposition 7.9 and the fact that a non-zero element of Λ has finitely many zeroes. \square

7.3 Case of bounded rank

Throughout this subsection, we will assume (B) and obtain formulas describing the growth of S_n along K_∞/K .

Definition 7.11. Assuming (B) (so that $L_p^\pm(E, K_\infty/K, X)$ is non-zero) define

$$\lambda^\pm = \lambda_E^\pm(K_\infty/K) = \lambda(L_p^\pm(E, K_\infty/K, X))$$

and

$$\mu^\pm = \mu_E^\pm(K_\infty/K) = \mu(L_p^\pm(E, K_\infty/K, X)).$$

We begin with a general lemma about the ‘‘growth’’ of torsion Λ -modules.

Lemma 7.12. *If Y is a torsion Λ -module, then for n large enough $\omega_{n-1}Y/\omega_nY$ is finite of size $\mu(Y) \cdot (p^n - p^{n-1}) + \lambda(Y) - \text{rk}_{\mathbf{Z}_p}(Y_{\Gamma_n})$.*

Proof. By the structure theory of Λ -modules, we may assume that Y is of the form Λ/f^e with f an irreducible polynomial. (Note that any finite groups that appear are killed by ω_n for n large enough.) If $\gcd(f, \omega_n) = 1$, then

$$\frac{\omega_{n-1}Y}{\omega_nY} \cong \frac{Y}{\xi_n Y} \cong \frac{\Lambda}{(f^e, \xi_n)} \cong \frac{\mathbf{Z}_p[\mu_{p^n}]}{f^e(\zeta_n - 1)}.$$

Now

$$\begin{aligned} \text{ord}_p \left(\# \frac{\mathbf{Z}_p[\mu_{p^n}]}{f^e(\zeta_n - 1)} \right) &= (p^n - p^{n-1}) \cdot \text{ord}_p(f^e(\zeta_n - 1)) \\ &= (p^n - p^{n-1}) \cdot \mu(f^e) + \lambda(f^e) \end{aligned}$$

which implies the lemma since $\text{rk}_{\mathbf{Z}_p}(Y_{\Gamma_n}) = 0$.

If $f = \xi_k$ for some $k \leq n$, then

$$\frac{\omega_{n-1}Y}{\omega_nY} \cong \frac{\Lambda}{(\xi_k^{e-1}, \xi_n)} \cong \frac{\mathbf{Z}_p[\mu_{p^n}]}{\xi_k^{e-1}(\zeta_n - 1)}$$

and the lemma follows since $\text{rk}_{\mathbf{Z}_p}(Y_{\Gamma_n}) = \deg(\xi_k)$. \square

The following duality theorem will be needed in what follows. Let

$$S_n^0(T) = \text{Sel}^0(T_p E/K_n) = \ker \left(S_n(T) \longrightarrow \prod_{\mathfrak{p}|p} E(K_{n,\mathfrak{p}}) \otimes \mathbf{Z}_p \right)$$

so that $S_{n,\Sigma}(T) \subseteq S_n^0(T) \subseteq S_n(T)$.

Theorem 7.13. *Let Y be the Λ -torsion submodule of \mathfrak{X}_∞ . Then assuming (W), Y is pseudo-isomorphic to $\text{Sel}_p^0(E[p^\infty]/K_\infty)^\wedge$. In particular, $\text{rk}_{\mathbf{Z}_p} Y_{\Gamma_n} = \text{rk}_{\mathbf{Z}_p} S_n^0(T) = \text{rk}_{\mathbf{Z}_p} S_{n,\Sigma}(T)$.*

Proof. The first statement is Corollary 2.5 in [26]. For the second statement, let $S_\infty^0 = \text{Sel}_p^0(E[p^\infty]/K_\infty)$ and $S_n^0 = \text{Sel}_p^0(E[p^\infty]/K_n)$. Then by [10, Remark 4.4], S_n^0 and $(S_\infty^0)^{\Gamma_n}$ differ only by finite groups. Therefore,

$$\text{rk}_{\mathbf{Z}_p} Y_{\Gamma_n} = \text{corank}_{\mathbf{Z}_p} (S_\infty^0)^{\Gamma_n} = \text{corank}_{\mathbf{Z}_p} S_n^0 = \text{rk}_{\mathbf{Z}_p} S_n^0(T).$$

Since, $E(K_{n,v}) \otimes \mathbf{Z}_p$ is finite for $v \nmid p$, we have that $\text{rk}_{\mathbf{Z}_p} S_n^0(T) = \text{rk}_{\mathbf{Z}_p} S_{n,\Sigma}(T)$ completing the proof. \square

Theorem 7.14. *Assuming (B), we have that*

$$\text{ord}_p(\#(S_n/S_{n-1})) = \begin{cases} \mu^+ \cdot (p^n - p^{n-1}) + (\lambda^+ - s) \cdot n + d \cdot q_n & 2|n \\ \mu^- \cdot (p^n - p^{n-1}) + (\lambda^- - s) \cdot n + d \cdot q_n & 2 \nmid n \end{cases}$$

where s is the stable value of $\text{corank}_{\mathbf{Z}_p} S_k$ and n is sufficiently large.

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{S_n(T)}{S_{n,\Sigma}(T)} & \longrightarrow & B_n \times \widehat{E}(K_{n,p}) & \xrightarrow{Q_n} & (\mathfrak{X}_\infty)_{\Gamma_n} & \longrightarrow & \mathfrak{X}_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \frac{S_{n-1}(T)}{S_{n-1,\Sigma}(T)} & \longrightarrow & B_{n-1} \times \widehat{E}(K_{n-1,p}) & \xrightarrow{Q_{n-1}} & (\mathfrak{X}_\infty)_{\Gamma_{n-1}} & \longrightarrow & \mathfrak{X}_{n-1} & \longrightarrow & 0 \end{array}$$

defined by Theorem 3.1. For n large enough, $S_n(T), S_{n,\Sigma}(T)$ and B_n stabilize and the vertical maps in the above diagram between these groups become multiplication by p .

We will break the above diagram into two pieces; namely

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{S_n(T)}{S_{n,\Sigma}(T)} & \longrightarrow & B_n \times \widehat{E}(K_{n,p}) & \longrightarrow & M_n & \longrightarrow & 0 \\ & & \cdot p \downarrow & & \cdot p \times \text{Tr}_{n-1}^n \downarrow & & \downarrow m_n & & (25) \\ 0 & \longrightarrow & \frac{S_{n-1}(T)}{S_{n-1,\Sigma}(T)} & \longrightarrow & B_{n-1} \times \widehat{E}(K_{n-1,p}) & \longrightarrow & M_{n-1} & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_n & \xrightarrow{Q_n} & (\mathfrak{X}_\infty)_{\Gamma_n} & \longrightarrow & \mathfrak{X}_n & \longrightarrow & 0 \\ & & m_n \downarrow & & \downarrow & & \downarrow \pi_n & & (26) \\ 0 & \longrightarrow & M_{n-1} & \xrightarrow{Q_{n-1}} & (\mathfrak{X}_\infty)_{\Gamma_{n-1}} & \longrightarrow & \mathfrak{X}_{n-1} & \longrightarrow & 0 \end{array}$$

where M_n is defined by the above diagrams. If $s_0 = \text{rk}_{\mathbf{Z}_p} S_{n,\Sigma}(T)$ and $h = \text{rk}_{\mathbf{F}_p} B_n/pB_n$, then applying the snake lemma to (25) yields

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\mathbf{Z}/p\mathbf{Z})^h \times \ker(\text{Tr}_{n-1}^n) & \longrightarrow & \ker(m_n) & \longrightarrow & (\mathbf{Z}/p\mathbf{Z})^{s-s_0} \longrightarrow \\ & & (\mathbf{Z}/p\mathbf{Z})^h \times \text{coker}(\text{Tr}_{n-1}^n) & \longrightarrow & \text{coker}(m_n) & \longrightarrow & 0. \end{array}$$

By Corollary 4.12, $\text{coker}(\text{Tr}_{n-1}^n) \cong (\mathbf{Z}/p\mathbf{Z})^{dq_n}$ and therefore we have that

$$\ker(m_n) \cong \ker(\text{Tr}_{n-1}^n) \times (\mathbf{Z}/p\mathbf{Z})^{h+a} \quad (27)$$

and

$$\text{coker}(m_n) \cong (\mathbf{Z}/p\mathbf{Z})^{dq_n+h-s+s_0+a} \quad (28)$$

for some a between 0 and $s - s_0$. Applying the snake lemma to (26) yields

$$0 \longrightarrow \ker(m_n) \longrightarrow \frac{\omega_{n-1}\mathfrak{X}_\infty}{\omega_n\mathfrak{X}_\infty} \longrightarrow (S_n/S_{n-1})^\wedge \longrightarrow \text{coker}(m_n) \longrightarrow 0. \quad (29)$$

For n large enough, S_n/S_{n-1} and $\frac{\omega_{n-1}Y}{\omega_n Y}$ are both finite and

$$\begin{aligned} \left[\frac{\omega_{n-1}\mathfrak{X}_\infty}{\omega_n\mathfrak{X}_\infty} : \ker(\text{Tr}_{n-1}^n) \right] &= \# \left(\frac{\omega_{n-1}Y}{\omega_n Y} \right) \cdot \left[\frac{\omega_{n-1}Z}{\omega_n Z} : \ker(\text{Tr}_{n-1}^n) \right] \\ &= \# \left(\frac{\omega_{n-1}Y}{\omega_n Y} \right) \cdot \left[\frac{\omega_{n-1}\Lambda}{\omega_n \Lambda} : \ker(\text{Tr}_{n-1}^n) \right] \\ &= \# \left(\frac{\omega_{n-1}Y}{\omega_n Y} \right) \cdot \# \left(\frac{(\Lambda/\xi_n)^d}{(u_1^n, \dots, u_d^n)} \right). \end{aligned}$$

Again, for n large enough, $L_p^\pm(E, K_\infty/K, \zeta_n - 1) \neq 0$ and hence $\det(u_{ij}^n(\zeta_n - 1)) \neq 0$. Also, by Lemma 7.8,

$$\text{ord}_p \left(\# \frac{(\Lambda/\xi_n)^d}{(u_1^n, \dots, u_d^n)} \right) = (\mu^\varepsilon - \mu_t) \cdot (p^n - p^{n-1}) + \lambda^\varepsilon - \lambda_t \quad (30)$$

where $\lambda_t = \lambda(t_Y)$, $\mu_t = \mu(t_Y)$ and $\varepsilon = (-1)^n$. Then, by Lemma 7.12, we have that

$$\text{ord}_p \left(\# \frac{\omega_{n-1}Y}{\omega_n Y} \right) = \mu_t \cdot (p^n - p^{n-1}) + \lambda_t - \text{rk}_{\mathbf{Z}_p}(Y_{\Gamma_n})$$

for n large enough. Thus,

$$\text{ord}_p \left[\frac{\omega_{n-1}\mathfrak{X}_\infty}{\omega_n\mathfrak{X}_\infty} : \ker(\text{Tr}_{n-1}^n) \right] = \mu^\varepsilon \cdot (p^n - p^{n-1}) + \lambda^\varepsilon - \text{rk}_{\mathbf{Z}_p}(Y_{\Gamma_n}).$$

Returning to (29), we can compute

$$\begin{aligned} \text{ord}_p(\#S_n/S_{n-1}) &= \text{ord}_p \left[\frac{\omega_{n-1}\mathfrak{X}_\infty}{\omega_n\mathfrak{X}_\infty} : \ker(m_n) \right] + \text{ord}_p(\# \text{coker}(m_n)) \\ &= -a - h + \text{ord}_p \left[\frac{\omega_{n-1}\mathfrak{X}_\infty}{\omega_n\mathfrak{X}_\infty} : \ker(\text{Tr}_{n-1}^n) \right] + \text{ord}_p(\# \text{coker}(m_n)) \\ &= \mu^\varepsilon \cdot (p^n - p^{n-1}) + \lambda^\varepsilon - \text{rk}_{\mathbf{Z}_p}(Y_{\Gamma_n}) + dq_n - s + s_0 \end{aligned}$$

Finally, from Theorem 7.13, we have that $s_0 = \text{rk}_{\mathbf{Z}_p}(Y_{\Gamma_n})$ which completes the proof of the theorem. \square

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