

KIDA'S FORMULA AND CONGRUENCES

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1. INTRODUCTION

Let f be a modular eigenform of weight at least two and let F be a finite abelian extension of \mathbf{Q} . Fix an odd prime p at which f is ordinary in the sense that the p^{th} Fourier coefficient of f is not divisible by p . In Iwasawa theory, one associates two objects to f over the cyclotomic \mathbf{Z}_p -extension F_∞ of F : a Selmer group $\text{Sel}(F_\infty, A_f)$ (where A_f denotes the divisible version of the two-dimensional Galois representation attached to f) and a p -adic L -function $L_p(F_\infty, f)$. In this paper we prove a formula, generalizing work of Kida and Hachimori–Matsuno, relating the Iwasawa invariants of these objects over F with their Iwasawa invariants over p -extensions of F .

For Selmer groups our results are significantly more general. Let T be a lattice in a nearly ordinary p -adic Galois representation V ; set $A = V/T$. When $\text{Sel}(F_\infty, A)$ is a cotorsion Iwasawa module, its Iwasawa μ -invariant $\mu^{\text{alg}}(F_\infty, A)$ is said to vanish if $\text{Sel}(F_\infty, A)$ is cofinitely generated and its λ -invariant $\lambda^{\text{alg}}(F_\infty, A)$ is simply its p -adic corank. We prove the following result relating these invariants in a p -extension.

Theorem 1. *Let F'/F be a finite Galois p -extension that is unramified at all places dividing p . Assume that T satisfies the technical assumptions (1)–(5) of Section 2. If $\text{Sel}(F_\infty, A)$ is Λ -cotorsion with $\mu^{\text{alg}}(F_\infty, A) = 0$, then $\text{Sel}(F'_\infty, A)$ is Λ -cotorsion with $\mu^{\text{alg}}(F'_\infty, A) = 0$. Moreover, in this case*

$$\lambda^{\text{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\text{alg}}(F_\infty, A) + \sum_{w'} m(F'_{\infty, w'} / F_{\infty, w}, V)$$

where the sum extends over places w' of F'_∞ which are ramified in F'_∞ / F_∞ .

If V is associated to a cuspform f and F' is an abelian extension of \mathbf{Q} , then the same results hold for the analytic Iwasawa invariants of f .

Here $m(F'_{\infty, w'} / F_{\infty, w}, V)$ is a certain difference of local multiplicities defined in Section 2.1. In the case of Galois representations associated to Hilbert modular forms, these local factors can be made quite explicit; see Section 4.1 for details.

It follows from Theorem 1 and work of Kato that if the p -adic main conjecture holds for a modular form f over \mathbf{Q} , then it holds for f over all abelian p -extensions of \mathbf{Q} ; see Section 4.2 for details.

These Riemann–Hurwitz type formulas were first discovered by Kida [5] in the context of λ -invariants of CM fields. More precisely, when F'/F is a p -extension of CM fields and $\mu^-(F_\infty/F) = 0$, Kida gave a precise formula for $\lambda^-(F'_\infty/F')$ in terms of $\lambda^-(F_\infty/F)$ and local data involving the primes that ramify in F'/F . (See also [4] for a representation theoretic interpretation of Kida's result.) A similar formula in a somewhat different setting was given for elliptic curves with complex multiplication at ordinary primes by Wingberg [12]; Hachimori–Matsuno [3] established

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the cyclotomic version in general. The analytic analogue was first established for ideal class groups by Sinnott [10] and for elliptic curves by Matsuno [7].

Our proof is most closely related to the arguments in [10] and [7] where congruences implicitly played a large role in their study of analytic λ -invariants. In this paper, we make the role of congruences more explicit and apply these methods to study both algebraic and analytic λ -invariants.

As is usual, we first reduce to the case where F'/F is abelian. (Some care is required to show that our local factors are well behaved in towers of fields; this is discussed in Section 2.1.) In this case, the λ -invariant of V over F' can be expressed as the sum of the λ -invariants of twists of V by characters of $\text{Gal}(F'/F)$. The key observation (already visible in both [10] and [7]) is that since $\text{Gal}(F'/F)$ is a p -group, all of its characters are trivial modulo a prime over p and, thus, the twisted Galois representations are all congruent to V modulo a prime over p . The algebraic case of Theorem 1 then follows from the results of [11] which gives a precise local formula for the difference between λ -invariants of congruent Galois representations. The analytic case is handled similarly using the results of [1].

The basic principle behind this argument is that a formula relating the Iwasawa invariants of congruent Galois representations should imply of a transition formula for these invariants in p -extensions. As an example of this, in Section 4.3, we use results of [2] to prove a Kida formula for the Iwasawa invariants (in the sense of [8, 6, 9]) of weight 2 modular forms at supersingular primes.

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2. ALGEBRAIC INVARIANTS

2.1. Local preliminaries. We begin by studying the local terms that appear in our results. Fix distinct primes ℓ and p and let L denote a finite extension of the cyclotomic \mathbf{Z}_p -extension of \mathbf{Q}_ℓ . Fix a field K of characteristic zero and a finite-dimensional K -vector space V endowed with a continuous K -linear action of the absolute Galois group G_L of L . Set

$$m_L(V) := \dim_K (V_{I_L})^{G_L},$$

the multiplicity of the trivial representation in the I_L -coinvariants of V . Note that this multiplicity is invariant under extension of scalars, so that we can enlarge K as necessary.

Let L' be a finite Galois p -extension of L . Note that L' must be cyclic and totally ramified since L contains the \mathbf{Z}_p -extension of \mathbf{Q}_ℓ . Let G denote the Galois group of L'/L . Assuming that K contains all $[L' : L]^{\text{th}}$ roots of unity, for a character $\chi : G \rightarrow K^\times$ of G , we set $V_\chi = V \otimes_K K(\chi)$ with $K(\chi)$ a one-dimensional K -vector space on which G acts via χ . We define

$$m(L'/L, V) := \sum_{\chi \in G^\vee} m_L(V) - m_L(V_\chi)$$

where G^\vee denotes the K -dual of G .

The next result shows how these invariants behave in towers of fields.

Lemma 2.1. *Let L'' be a finite Galois p -extension of L and let L' be a Galois extension of L contained in L'' . Assume that K contains all $[L'' : L]^{\text{th}}$ roots of*

unity. Then

$$m(L''/L, V) = [L'' : L'] \cdot m(L'/L, V) + m(L''/L', V).$$

Proof. Set $G = \text{Gal}(L''/L)$ and $H = \text{Gal}(L''/L')$. Consider the Galois group $G_L/I_{L''}$ over L of the maximal unramified extension of L'' . It sits in an exact sequence

$$(1) \quad 0 \rightarrow G_{L''}/I_{L''} \rightarrow G_L/I_{L''} \rightarrow G \rightarrow 0$$

which is in fact split since the maximal unramified extensions of both L and L'' are obtained by adjoining all prime-to- p roots of unity.

Fix a character $\chi \in G^\vee$. We compute

$$\begin{aligned} m_L(V_\chi) &= \dim_K((V_\chi)_{I_L})^{G_L} \\ &= \dim_K\left(\left(\left(\left(V_\chi\right)_{I_{L''}}\right)_G\right)^{G_{L''}}\right)^G \\ &= \dim_K\left(\left(\left(\left(V_\chi\right)_{I_{L''}}\right)^{G_{L''}}\right)_G\right)^G \quad \text{since (1) is split} \\ &= \dim_K\left(\left(\left(V_\chi\right)_{I_{L''}}\right)^{G_{L''}}\right)^G \quad \text{since } G \text{ is finite cyclic} \\ &= \dim_K\left(\left(V_{I_{L''}}\right)^{G_{L''}} \otimes \chi\right)^G \quad \text{since } \chi \text{ is trivial on } G_{L''}. \end{aligned}$$

The lemma thus follows from the following purely group-theoretical statement applied with $W = (V_{I_{L''}})^{G_{L''}}$: for a finite dimensional representation W of a finite abelian group G over a field of characteristic zero containing $\mu_{\#G}$, we have

$$\begin{aligned} \sum_{\chi \in G^\vee} (\langle W, 1 \rangle_G - \langle W, \chi \rangle_G) &= \\ \#H \cdot \sum_{\chi \in (G/H)^\vee} (\langle W, 1 \rangle_G - \langle W, \chi \rangle_G) &+ \sum_{\chi \in H^\vee} (\langle W, 1 \rangle_H - \langle W, \chi \rangle_H) \end{aligned}$$

for any subgroup H of G ; here $\langle W, \chi \rangle_G$ (resp. $\langle W, \chi \rangle_H$) is the multiplicity of the character χ in W regarded as a representation of G (resp. H). To prove this, we compute

$$\begin{aligned} &\sum_{\chi \in G^\vee} (\langle W, 1 \rangle_G - \langle W, \chi \rangle_G) \\ &= \#G \cdot \langle W, 1 \rangle_G - \left\langle W, \text{Ind}_1^G 1 \right\rangle_G \\ &= \#G \cdot \langle W, 1 \rangle_G - \#H \cdot \left\langle W, \text{Ind}_H^G 1 \right\rangle_G + \#H \cdot \left\langle W, \text{Ind}_H^G 1 \right\rangle_G - \left\langle W, \text{Ind}_1^G 1 \right\rangle_G \\ &= \#H \cdot \sum_{\chi \in (G/H)^\vee} (\langle W, 1 \rangle_G - \langle W, \chi \rangle_G) + \sum_{\chi \in H^\vee} \left(\left\langle W, \text{Ind}_H^G 1 \right\rangle_G - \left\langle W, \text{Ind}_H^G \chi \right\rangle_G \right) \\ &= \#H \cdot \sum_{\chi \in (G/H)^\vee} (\langle W, 1 \rangle_G - \langle W, \chi \rangle_G) + \sum_{\chi \in H^\vee} (\langle W, 1 \rangle_H - \langle W, \chi \rangle_H) \end{aligned}$$

by Frobenius reciprocity. \square

2.2. Global preliminaries. Fix a number field F ; for simplicity we assume that F is either totally real or totally imaginary. Fix also an odd prime p and a finite extension K of \mathbf{Q}_p ; we write \mathcal{O} for the ring of integers of K , π for a fixed choice of uniformizer of \mathcal{O} , and $k = \mathcal{O}/\pi$ for the residue field of \mathcal{O} .

Let T be a nearly ordinary Galois representation over F with coefficients in \mathcal{O} ; that is, T is a free \mathcal{O} -module of some rank n endowed with an \mathcal{O} -linear action of the absolute Galois group G_F , together with a choice for each place v of F dividing p of a complete flag

$$0 = T_v^0 \subset T_v^1 \subset \cdots \subset T_v^n = T$$

stable under the action of the decomposition group $G_v \subseteq G_F$ of v . We make the following assumptions on T :

- (1) For each place v dividing p we have

$$(T_v^i/T_v^{i-1}) \otimes k \not\cong (T_v^j/T_v^{j-1}) \otimes k$$

as $k[G_v]$ -modules for all $i \neq j$;

- (2) If F is totally real, then $\text{rank } T^{c_v=1}$ is independent of the archimedean place v (here c_v is a complex conjugation at v);
(3) If F is totally imaginary, then n is even.

Remark 2.2. The conditions above are significantly more restrictive than are actually required to apply the results of [11]. As our main interest is in abelian (and thus necessarily Galois) extensions of \mathbf{Q} , we have chosen to include the assumptions (2) and (3) to simplify the exposition. The assumption (1) is also stronger than necessary: all that is actually needed is that the centralizer of $T \otimes k$ consists entirely of scalars and that $\mathfrak{gl}_n/\mathfrak{b}_v$ has trivial adjoint G_v -invariants for all places v dividing p ; here \mathfrak{gl}_n denotes the p -adic Lie algebra of GL_n and \mathfrak{b}_v denotes the p -adic Lie algebra of the Borel subgroup associated to the complete flag at v . In particular, when T has rank 2, we may still allow the case that $T \otimes k$ has the form

$$\begin{pmatrix} \chi & * \\ 0 & \chi \end{pmatrix}$$

so long as $*$ is non-trivial. (Equivalently, if T is associated to a modular form f , the required assumption is that f is p -distinguished.)

Set $A = T \otimes_{\mathcal{O}} K/\mathcal{O}$; it is a cofree \mathcal{O} -module of corank n with an \mathcal{O} -linear action of G_F . Let c equal the rank of $T^{c_v=1}$ (resp. $n/2$) if F is totally real (resp. totally imaginary) and set

$$A_v^{\text{cr}} := \text{im}(T_v^c \otimes_{\mathcal{O}} K \hookrightarrow T \otimes_{\mathcal{O}} K \rightarrow A).$$

We define the Selmer group of A over the cyclotomic \mathbf{Z}_p -extension F_{∞} of F by

$$\text{Sel}(F_{\infty}, A) = \ker \left(H^1(F_{\infty}, A) \rightarrow \left(\bigoplus_{w \nmid p} H^1(F_{\infty, w}, A) \right) \times \left(\bigoplus_{w \mid p} H^1(F_{\infty, w}, A/A_v^{\text{cr}}) \right) \right).$$

The Selmer group $\text{Sel}(F_{\infty}, A)$ is naturally a module for the Iwasawa algebra $\Lambda_{\mathcal{O}} := \mathcal{O}[[\text{Gal}(F_{\infty}/F)]]$. If $\text{Sel}(F_{\infty}, A)$ is $\Lambda_{\mathcal{O}}$ -cotorsion (that is, if the dual of $\text{Sel}(F_{\infty}, A)$ is a torsion $\Lambda_{\mathcal{O}}$ -module), then we write $\mu^{\text{alg}}(F_{\infty}, A)$ and $\lambda^{\text{alg}}(F_{\infty}, A)$ for its Iwasawa invariants; in particular, $\mu^{\text{alg}}(F_{\infty}, A) = 0$ if and only if $\text{Sel}(F_{\infty}, A)$ is a cofinitely generated \mathcal{O} -module, while $\lambda^{\text{alg}}(F_{\infty}, A)$ is the \mathcal{O} -corank of $\text{Sel}(F_{\infty}, A)$.

Remark 2.3. In the case that T is in fact an *ordinary* Galois representation (meaning that the action of inertia on each T_v^i/T_v^{i-1} is by an integer power e_i (independent of v) of the cyclotomic character such that $e_1 > e_2 > \dots > e_n$), then our Selmer group $\text{Sel}(F_\infty, A)$ is simply the Selmer group in the sense of Greenberg of a twist of A ; see [11, Section 1.3] for details.

2.3. Extensions. Let F' be a finite Galois extension of F with degree equal to a power of p . We write F'_∞ for the cyclotomic \mathbf{Z}_p -extension of F' and set $G = \text{Gal}(F'_\infty/F_\infty)$. Note that T satisfies hypotheses (1)–(3) over F' as well, so that we may define $\text{Sel}(F'_\infty, A)$ analogously to $\text{Sel}(F_\infty, A)$. (For (1) this follows from the fact that G_v acts on $(T_v^i/T_v^{i-1}) \otimes k$ by a character of prime-to- p order; for (2) and (3) it follows from the fact that p is assumed to be odd.)

Lemma 2.4. *The restriction map*

$$(2) \quad \text{Sel}(F_\infty, A) \rightarrow \text{Sel}(F'_\infty, A)^G$$

has finite kernel and cokernel.

Proof. This is straightforward from the definitions and the fact that G is finite and A is cofinitely generated; see [3, Lemma 3.3] for details. \square

We can use Lemma 2.4 to relate the μ -invariants of A over F_∞ and F'_∞ .

Corollary 2.5. *If $\text{Sel}(F_\infty, A)$ is Λ -cotorsion with $\mu^{\text{alg}}(F_\infty, A) = 0$, then $\text{Sel}(F'_\infty, A)$ is Λ -cotorsion with $\mu^{\text{alg}}(F'_\infty, A) = 0$.*

Proof. This is a straightforward argument using Lemma 2.4 and Nakayama's lemma for compact local rings; see [3, Corollary 3.4] for details. \square

Fix a finite extension K' of K containing all $[F' : F]^{\text{th}}$ roots of unity. Consider a character $\chi : G \rightarrow \mathcal{O}'^\times$ taking values in the ring of integers \mathcal{O}' of K' ; note that χ is necessarily even since $[F' : F]$ is odd. We set

$$A_\chi = A \otimes_{\mathcal{O}} \mathcal{O}'(\chi)$$

where $\mathcal{O}'(\chi)$ is a free \mathcal{O}' -module of rank one with G_{F_∞} -action given by χ . If we give A_χ the induced complete flags at places dividing p , then A_χ satisfies hypotheses (1)–(3) and we have

$$A_{\chi, v}^{\text{cr}} = A_v^{\text{cr}} \otimes_{\mathcal{O}} \mathcal{O}'(\chi) \subseteq A_\chi$$

for each place v dividing p . We write $\text{Sel}(F_\infty, A_\chi)$ for the corresponding Selmer group, regarded as a $\Lambda_{\mathcal{O}'}$ -module; in particular, by $\lambda^{\text{alg}}(F_\infty, A_\chi)$ we mean the \mathcal{O}' -corank of $\text{Sel}(F_\infty, A_\chi)$, rather than the \mathcal{O} -corank. We write G^\vee for the set of all characters $\chi : G \rightarrow \mathcal{O}'^\times$.

Proposition 2.6. *Assume that $\text{Sel}(F_\infty, A)$ is Λ -cotorsion with $\mu^{\text{alg}}(F_\infty, A) = 0$. If G is an abelian group, then there is a natural map*

$$\bigoplus_{\chi \in G^\vee} \text{Sel}(F_\infty, A_\chi) \rightarrow \text{Sel}(F'_\infty, A) \otimes_{\mathcal{O}} \mathcal{O}'$$

with finite kernel and cokernel.

Proof. First note that as $\mathcal{O}'[[G_{F'}]]$ -modules we have

$$A \otimes_{\mathcal{O}} \mathcal{O}' \cong A_\chi$$

from which it easily follows that

$$(3) \quad (\mathrm{Sel}(F'_\infty, A) \otimes_{\mathcal{O}} \mathcal{O}'(\chi))^G = \mathrm{Sel}(F'_\infty, A_\chi)^G.$$

Also, for any cofinitely generated $\mathcal{O}[G]$ -module S , the natural map

$$(4) \quad \bigoplus_{\chi \in G^\vee} (S \otimes \mathcal{O}'(\chi))^G \rightarrow S \otimes \mathcal{O}'$$

has finite kernel and cokernel. Since we are assuming that $\mu^{\mathrm{alg}}(F_\infty, A) = 0$, we may take $S = \mathrm{Sel}(F'_\infty, A)$ in (4); combining this with (3) yields a map

$$\bigoplus_{\chi \in G^\vee} (\mathrm{Sel}(F'_\infty, A_\chi))^G \rightarrow \mathrm{Sel}(F'_\infty, A_\chi) \otimes \mathcal{O}'$$

with finite kernel and cokernel. Now applying Lemma 2.4 for each twist A_χ , we obtain our proposition. \square

As an immediate corollary, we have the following.

Corollary 2.7. *If $\mathrm{Sel}(F_\infty, A)$ is Λ -cotorsion with $\mu^{\mathrm{alg}}(F_\infty, A) = 0$, then each group $\mathrm{Sel}(F_\infty, A_\chi)$ is $\Lambda_{\mathcal{O}'}$ -cotorsion with $\mu^{\mathrm{alg}}(F_\infty, A_\chi) = 0$. Moreover, if G is abelian, then*

$$\lambda^{\mathrm{alg}}(F'_\infty, A) = \sum_{\chi \in G^\vee} \lambda^{\mathrm{alg}}(F_\infty, A_\chi).$$

2.4. Algebraic transition formula. We continue with the notation of the previous section. We write $R(F'_\infty/F_\infty)$ for the set of prime-to- p places of F'_∞ which are ramified in F'_∞/F_∞ . For a place $w' \in R(F'_\infty/F_\infty)$, we write w for its restriction to F_∞ .

Theorem 2.8. *Let F'/F be a finite Galois p -extension with Galois group G which is unramified at all places dividing p . Let T be a nearly ordinary Galois representation over F with coefficients in \mathcal{O} satisfying (1)–(3). Set $A = T \otimes K/\mathcal{O}$ and assume that:*

- (4) $H^0(F, A[\pi]) = H^0(F, \mathrm{Hom}(A[\pi], \mu_p)) = 0$;
- (5) $H^0(I_v, A/A_v^{cr})$ is \mathcal{O} -divisible for all v dividing p .

If $\mathrm{Sel}(F_\infty, A)$ is Λ -cotorsion with $\mu^{\mathrm{alg}}(F_\infty, A) = 0$, then $\mathrm{Sel}(F'_\infty, A)$ is Λ -cotorsion with $\mu^{\mathrm{alg}}(F'_\infty, A) = 0$. Moreover, in this case,

$$\lambda^{\mathrm{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\mathrm{alg}}(F_\infty, A) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty, w'}/F_{\infty, w}, V)$$

with $V = T \otimes K$ and $m(F'_{\infty, w'}/F_{\infty, w}, V)$ as in Section 2.1.

Note that $m(F'_{\infty, w'}/F_{\infty, w}, V)$ in fact depends only on w and not on w' . The hypotheses (4) and (5) are needed to apply the results of [11]; they will not otherwise appear in the proof below. We note that the assumption that F'/F is unramified at p is primarily needed to assure that the condition (5) holds for twists of A as well.

Since p -groups are solvable and the only simple p -group is cyclic, the next lemma shows that it suffices to consider the case of $\mathbf{Z}/p\mathbf{Z}$ -extensions.

Lemma 2.9. *Let F''/F be a Galois p -extension of number fields and let F' be an intermediate extension which is Galois over F . Let T be as above. If Theorem 2.8 holds for T with respect to any two of the three field extensions F''/F' , F'/F and F''/F , then it holds for T with respect to the third extension.*

Proof. This is clear from Corollary 2.5 except for the λ -invariant formula. Substituting the formula for $\lambda(F'_\infty, A)$ in terms of $\lambda(F_\infty, A)$ into the formula for $\lambda(F''_\infty, A)$ in terms of $\lambda(F'_\infty, A)$, one finds that it suffices to show that

$$\begin{aligned} \sum_{w'' \in R(F''_\infty/F_\infty)} m(F''_{\infty, w''}/F_{\infty, w}, V) = \\ [F''_\infty : F'_\infty] \cdot \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty, w'}/F_{\infty, w}, V) \\ + \sum_{w'' \in R(F''_\infty/F'_\infty)} m(F''_{\infty, w''}/F'_{\infty, w'}, V). \end{aligned}$$

This formula follows upon summing the formula of Lemma 2.1 over all $w'' \in R(F''_\infty/F_\infty)$ and using the two facts:

- $[F''_\infty : F'_\infty]/[F''_{\infty, w''} : F'_{\infty, w'}]$ equals the number of places of F''_∞ lying over w' (since the residue field of $F_{\infty, w}$ has no p -extensions);
- $m(F''_{\infty, w''}/F'_{\infty, w'}, V) = 0$ for any $w'' \in R(F''_\infty/F_\infty) - R(F''_\infty/F'_\infty)$.

□

Proof of Theorem 2.8. By Lemma 2.9 and the preceding remark, we may assume that F'_∞/F_∞ is a cyclic extension of degree p . The fact that $\text{Sel}(F'_\infty, A)$ is cotorsion with trivial μ -invariant is simply Corollary 2.5. Furthermore, by Corollary 2.7, we have

$$\lambda^{\text{alg}}(F'_\infty, A) = \sum_{\chi \in G^\vee} \lambda^{\text{alg}}(F_\infty, A_\chi).$$

For $\chi \in G^\vee$, note that χ is trivial modulo a uniformizer π' of \mathcal{O}' as it takes values in μ_p . In particular, the residual representations $A_\chi[\pi']$ and $A[\pi]$ are isomorphic. Under the hypotheses (1)–(5), the result [11, Theorem 1] gives a precise formula for the relation between λ -invariants of congruent Galois representations. In the present case it takes the form:

$$\lambda^{\text{alg}}(F_\infty, A_\chi) = \lambda^{\text{alg}}(F_\infty, A) + \sum_{w' \nmid p} (m_{F_{\infty, w}}(V \otimes \omega^{-1}) - m_{F_{\infty, w}}(V_\chi \otimes \omega^{-1}))$$

where the sum is over all prime-to- p places w' of F'_∞ , w denotes the place of F_∞ lying under w' and ω is the mod p cyclotomic character. The only non-zero terms in this sum are those for which w' is ramified in F'_∞/F_∞ . For any such w' , we have $\mu_p \subseteq F_{\infty, w}$ by local class field theory so that ω is in fact trivial at w ; thus

$$\lambda^{\text{alg}}(F_\infty, A_\chi) = \lambda^{\text{alg}}(F_\infty, A) + \sum_{w' \in R(F'_\infty/F_\infty)} (m_{F_{\infty, w}}(V) - m_{F_{\infty, w}}(V_\chi)).$$

Summing over all $\chi \in G^\vee$ then yields

$$\lambda^{\text{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\text{alg}}(F_\infty, A) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty, w'}/F_{\infty, w}, V)$$

which completes the proof. □

3. ANALYTIC INVARIANTS

3.1. Definitions. Let $f = \sum a_n q^n$ be a modular eigenform of weight $k \geq 2$, level N and character ε . Let K denote the finite extension of \mathbf{Q}_p generated by the Fourier coefficients of f (under some fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$), let \mathcal{O} denote the ring of integers of K and let k denote the residue field of \mathcal{O} . Let V_f denote a two-dimensional K -vector space with Galois action associated to f in the usual way; thus the characteristic polynomial of a Frobenius element at a prime $\ell \nmid Np$ is

$$x^2 - a_\ell x + \ell^{k-1} \varepsilon(\ell).$$

Fix a Galois stable \mathcal{O} -lattice T_f in V_f . We assume that $T_f \otimes k$ is an irreducible Galois representation; in this case T_f is uniquely determined up to scaling. Set $A_f = T_f \otimes K/\mathcal{O}$.

Assuming that f is p -ordinary (in the sense that a_p is relatively prime to p) and fixing a canonical period for f , one can associate to f a p -adic L -function $L_p(\mathbf{Q}_\infty/\mathbf{Q}, f)$ which lies in $\Lambda_{\mathcal{O}}$. This is well-defined up to a p -adic unit (depending upon the choice of a canonical period) and thus has well-defined Iwasawa invariants.

Let F/\mathbf{Q} be a finite abelian extension and let F_∞ denote the cyclotomic \mathbf{Z}_p -extension of F . For a character χ of $\text{Gal}(F/\mathbf{Q})$, we denote by f_χ the modular eigenform $\sum a_n \chi(n) q^n$ obtained from f by twisting by χ (viewed as a Dirichlet character). If f is p -ordinary and F/\mathbf{Q} is unramified at p , then f_χ is again p -ordinary and we define

$$L_p(F_\infty/F, f) = \prod_{\chi \in \text{Gal}(F/\mathbf{Q})^\vee} L_p(\mathbf{Q}_\infty/\mathbf{Q}, f_\chi).$$

If F/\mathbf{Q} is ramified at p , it is still possible to define $L_p(F_\infty/F, f)$; see [7, pg. 5], for example.

If F_1 and F_2 are two distinct number fields whose cyclotomic \mathbf{Z}_p -extensions agree, the corresponding p -adic L -functions of f over F_1 and F_2 need not agree. However, it is easy to check that the λ -invariants of these two power series are equal while their μ -invariants differ by a factor of a power of p . As we are only interested in the case of vanishing μ -invariants, we will abuse notation somewhat and simply denote the Iwasawa invariants of $L_p(F_\infty/F, f)$ by $\mu^{\text{an}}(F_\infty, f)$ and $\lambda^{\text{an}}(F_\infty, f)$.

3.2. Analytic transition formula. Let F/\mathbf{Q} be a finite abelian extension of \mathbf{Q} and let F' be a finite p -extension of F such that F'/\mathbf{Q} is abelian. As always, let F_∞ and F'_∞ denote the cyclotomic \mathbf{Z}_p -extensions of F and F' . As before, we write $R(F'_\infty/F_\infty)$ for the set of prime-to- p places of F'_∞ which are ramified in F'_∞/F_∞ .

Theorem 3.1. *Let f be a p -ordinary modular form such that $T_f \otimes k$ is irreducible and p -distinguished. If $\mu^{\text{an}}(F_\infty, f) = 0$, then $\mu^{\text{an}}(F'_\infty, f) = 0$. Moreover, if this is the case, then*

$$\lambda^{\text{an}}(F'_\infty, f) = [F'_\infty : F_\infty] \cdot \lambda^{\text{an}}(F_\infty, f) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty, w'}/F_{\infty, w}, V_f).$$

Proof. By Lemma 2.9, we may assume $[F : \mathbf{Q}]$ is prime-to- p . Indeed, let F_0 be the maximal subfield of F of prime-to- p degree over \mathbf{Q} . By Lemma 2.9, knowledge of the theorem for the two extensions F'/F_0 and F/F_0 would then imply it for F'/F as well. Furthermore, replacing F (resp. F') by the maximal tamely ramified subextension of F_∞ (resp. F'_∞), we may assume that every character of $\text{Gal}(F/\mathbf{Q})$

and $\text{Gal}(F'/\mathbf{Q})$ is the product of a power of the mod p cyclotomic character and a character unramified at p .

After making these reductions, we let M denote the (unique) p -extension of \mathbf{Q} inside of F' such that $MF = F'$. Set $G = \text{Gal}(F/\mathbf{Q})$ and $H = \text{Gal}(M/\mathbf{Q})$, so that $\text{Gal}(F'/\mathbf{Q}) \cong G \times H$. We have

$$(5) \quad \mu^{\text{an}}(F_\infty, f) = \sum_{\psi \in \text{Gal}(F/\mathbf{Q})^\vee} \mu^{\text{an}}(\mathbf{Q}_\infty, f_\psi)$$

and

$$(6) \quad \mu^{\text{an}}(F'_\infty, f) = \sum_{\psi \in \text{Gal}(F'/\mathbf{Q})^\vee} \mu^{\text{an}}(\mathbf{Q}_\infty, f_\psi) = \sum_{\psi \in G^\vee} \sum_{\chi \in H^\vee} \mu^{\text{an}}(\mathbf{Q}_\infty, f_{\psi\chi}).$$

Since we are assuming that $\mu^{\text{an}}(F_\infty, f) = 0$ and since these μ -invariants are non-negative, from (5) it follows that $\mu^{\text{an}}(\mathbf{Q}_\infty, f_\psi) = 0$ for each $\psi \in \text{Gal}(F/\mathbf{Q})^\vee$.

Fix $\psi \in G^\vee$. For any $\chi \in H^\vee$, $\psi\chi$ is congruent to ψ modulo any prime over p and thus f_χ and $f_{\psi\chi}$ are congruent modulo any prime over p . Then, since $\mu^{\text{an}}(\mathbf{Q}_\infty, f_\psi) = 0$, by [1, Theorem 3.7.5] it follows that $\mu^{\text{an}}(\mathbf{Q}_\infty, f_{\psi\chi}) = 0$ for each $\chi \in H^\vee$. (Note that the results of [1] apply to twists of p -ordinary forms by powers of the mod p cyclotomic character; this is why the reduction to the tamely ramified case is necessary for this argument.) Therefore, by (6) we have that $\mu^{\text{an}}(F'_\infty, f) = 0$ proving the first part of the theorem.

For λ -invariants, we again have

$$\lambda^{\text{an}}(F_\infty, f) = \sum_{\psi \in \text{Gal}(F/\mathbf{Q})^\vee} \lambda^{\text{an}}(\mathbf{Q}_\infty, f_\psi).$$

and

$$(7) \quad \lambda^{\text{an}}(F'_\infty, f) = \sum_{\psi \in G^\vee} \sum_{\chi \in H^\vee} \lambda^{\text{an}}(\mathbf{Q}_\infty, f_{\psi\chi}).$$

By [1, Theorem 3.7.7] the congruence between f_χ and $f_{\psi\chi}$ implies that

$$\lambda^{\text{an}}(\mathbf{Q}_\infty, f_{\psi\chi}) - \lambda^{\text{an}}(\mathbf{Q}_\infty, f_\psi) = \sum_v (m_{\mathbf{Q}_{\infty,v}}(V_{f_{\psi\chi}} \otimes \omega^{-1}) - m_{\mathbf{Q}_{\infty,v}}(V_{f_\psi} \otimes \omega^{-1}))$$

where the sum is over all places v of \mathbf{Q}_∞ at which χ is ramified. (Note that in [1] the sum extends over all prime-to- p places; however, the terms are trivial unless χ is ramified at v . Also note that the mod p cyclotomic characters that appear are actually trivial since if $\mathbf{Q}_{\infty,v}$ has a ramified Galois p -extensions for $v \nmid p$, then $\mu_p \subseteq \mathbf{Q}_{\infty,v}$.)

Combining this with (7) and the definition of $m(M_{\infty, v'} / \mathbf{Q}_{\infty, v}, V_{f_{\psi}})$, we conclude that

$$\begin{aligned} \lambda^{\text{an}}(F'_{\infty}, f) &= \sum_{\psi \in G^{\vee}} \left([F'_{\infty} : F_{\infty}] \cdot \lambda^{\text{an}}(\mathbf{Q}_{\infty}, f_{\psi}) + \sum_{v' \in R(M_{\infty} / \mathbf{Q}_{\infty})} m(M_{\infty, v'} / \mathbf{Q}_{\infty, v}, V_{f_{\psi}}) \right) \\ &= [F'_{\infty} : F_{\infty}] \cdot \lambda^{\text{an}}(F_{\infty}, f) + \sum_{v' \in R(M_{\infty} / \mathbf{Q}_{\infty})} \sum_{\psi \in G^{\vee}} m(M_{\infty, v'} / \mathbf{Q}_{\infty, v}, V_{f_{\psi}}) \\ &= [F'_{\infty} : F_{\infty}] \cdot \lambda^{\text{an}}(F_{\infty}, f) + \sum_{v' \in R(M_{\infty} / \mathbf{Q}_{\infty})} g_{v'}(F'_{\infty} / M_{\infty}) \cdot \\ &\quad m(M_{\infty, v'} / \mathbf{Q}_{\infty, v}, \mathbf{Z}[\text{Gal}(F_{\infty, w} / \mathbf{Q}_{\infty, v})] \otimes V_f) \end{aligned}$$

where $g_{v'}(F'_{\infty} / M_{\infty})$ denotes the number of places of F'_{∞} above the place v' of M_{∞} . By Frobenius reciprocity,

$$m(M_{\infty, v'} / \mathbf{Q}_{\infty, v}, \mathbf{Z}[\text{Gal}(F_{\infty, w} / \mathbf{Q}_{\infty, v})] \otimes V_f) = m(F'_{\infty, w'} / F_{\infty, w}, V_f)$$

where w' is the unique place of F'_{∞} above v' and w . It follows that

$$\lambda(F'_{\infty}, f) = [F'_{\infty} : F_{\infty}] \cdot \lambda^{\text{an}}(F_{\infty}, f) + \sum_{w' \in R(F'_{\infty} / F_{\infty})} m(F'_{\infty, w'} / F_{\infty, w}, V_f)$$

as desired. \square

4. ADDITIONAL RESULTS

4.1. Hilbert modular forms. We illustrate our results in the case of the two-dimensional representation V_f associated to a Hilbert modular eigenform f over a totally real field F . Although in principle our analytic results should remain true in this context, we focus on the less conjectural algebraic picture. Fix a G_F -stable lattice $T_f \subseteq V_f$ and let $A_f = T_f \otimes K / \mathcal{O}$.

Let F' be a finite Galois p -extension of F unramified at all places dividing p ; for simplicity we assume also that F' is linearly disjoint from F_{∞} . Let v be a place of F not dividing p and fix a place v' of F' lying over v . For a character φ of G_v , we define

$$h(\varphi) = \begin{cases} -1 & \varphi \text{ ramified, } \varphi|_{G_{v'}} \text{ unramified, and } \varphi \equiv 1 \pmod{\pi}; \\ 0 & \varphi \not\equiv 1 \pmod{\pi} \text{ or } \varphi|_{G_{v'}} \text{ ramified}; \\ e_v(F'/F) - 1 & \varphi \text{ unramified and } \varphi \equiv 1 \pmod{\pi} \end{cases}$$

where $e_v(F'/F)$ denotes the ramification index of v in F'/F and $G_{v'}$ is the decomposition group at v' . Set

$$h_v(f) = \begin{cases} h(\varphi_1) + h(\varphi_2) & f \text{ principal series with characters } \varphi_1, \varphi_2 \text{ at } v; \\ h(\varphi) & f \text{ special with character } \varphi \text{ at } v; \\ 0 & f \text{ supercuspidal or extraordinary at } v. \end{cases}$$

For example, if f is unramified principal series at v with Frobenius characteristic polynomial

$$x^2 - a_v x + c_v,$$

then

$$h_v(f) = \begin{cases} 2(e_v(F'/F) - 1) & a_v \equiv 2, c_v \equiv 1 \pmod{\pi} \\ e_v(F'/F) - 1 & a_v \equiv c_v + 1 \not\equiv 2 \pmod{\pi} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.1. *Assume that f is ordinary (in the sense that for each place v dividing p the Galois representation V_f has a unique one-dimensional quotient unramified at v) and that*

$$H^0(F, A_f[\pi]) = H^0(F, \text{Hom}(A_f[\pi], \mu_p)) = 0.$$

If $\text{Sel}(F_\infty, A_f)$ is Λ -cotorsion with $\mu^{\text{alg}}(F_\infty, A_f) = 0$, then also $\text{Sel}(F'_\infty, A_f)$ is Λ -cotorsion with $\mu^{\text{alg}}(F'_\infty, A_f) = 0$ and

$$\lambda^{\text{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\text{alg}}(F_\infty, A) + \sum_v g_v(F'_\infty/F) \cdot h_v(f);$$

here the sum is over the prime-to- p places of F ramified in F'_∞ and $g_v(F'_\infty/F)$ denotes the number of places of F'_∞ lying over such a v .

Proof. Fix a place v of F not dividing p and let w denote a place of F_∞ lying over v . Since there are exactly $g_v(F_\infty/F)$ such places, by Theorem 2.8 it suffices to prove that

$$(8) \quad h_v(f) = m(F'_{\infty, w'}/F_{\infty, w}, V_f) := \sum_{\chi \in \text{Gal}(F'_{\infty, w'}/F_{\infty, w})^\vee} (m_{F_{\infty, w}}(V_f) - m_{F_{\infty, w}}(V_{f, \chi})).$$

This is a straightforward case analysis. We will discuss the case that V_f is special associated to a character φ at v ; the other cases are similar. In the special case, we have

$$V_{f, \chi}|_{I_{F_{\infty, w}}} = \begin{cases} K'(\chi\varphi) & \chi\varphi|_{G_{F_{\infty, w}}} \text{ unramified;} \\ 0 & \chi\varphi|_{G_{F_{\infty, w}}} \text{ ramified.} \end{cases}$$

Since an unramified character has trivial restriction to $G_{F_{\infty, w}}$ if and only if it has trivial reduction modulo π , it follows that

$$m_{F_{\infty, w}}(V_{f, \chi}) = \begin{cases} 1 & \varphi \equiv 1 \pmod{\pi} \text{ and } \chi\varphi|_{G_{F_{\infty, w}}} \text{ unramified;} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the sum in (8) is zero if $\varphi \not\equiv 1 \pmod{\pi}$ or if φ is ramified when restricted to $G_{F'_{\infty, w'}}$ (as then $\chi\varphi$ is ramified for all $\chi \in G_v^\vee$). If $\varphi \equiv 1 \pmod{\pi}$ and φ itself is unramified, then $m_{F_{\infty, w}}(V_f) = 1$ while $m_{F_{\infty, w}}(V_{f, \chi}) = 0$ for $\chi \neq 1$, so that the sum in (8) is $[F'_{\infty, w'} : F_{\infty, w}] - 1 = e_v(F'/F) - 1$, as desired. Finally, if $\varphi \equiv 1 \pmod{\pi}$ and φ is ramified but becomes unramified when restricted to $G_{v'}$, then $m_{F_{\infty, w}}(V_f) = 0$, while $m_{F_{\infty, w}}(V_{f, \chi}) = 1$ for a unique χ , so that the sum is -1 . \square

Suppose finally that f is in fact the Hilbert modular form associated to an elliptic curve E over F . The only principal series which occur are unramified and we have $c_v \equiv 1 \pmod{\pi}$ (since the determinant of V_f is cyclotomic and F_∞ has a p -extension (namely, F'_∞) ramified at v), so that

$$h_v(f) \neq 0 \Leftrightarrow a_v \equiv 2 \Leftrightarrow E(F_v) \text{ has a point of order } p$$

in which case $h_v(f) = 2(e_v(F'/F) - 1)$. The only characters which may occur in a special constituent are trivial or unramified quadratic, and we have $h_v(f) =$

$e_v(F'/F) - 1$ or 0 respectively. Thus Theorem 4.1 recovers [3, Theorem 3.1] in this case.

4.2. The main conjecture. Let f be a p -ordinary elliptic modular eigenform of weight at least two and arbitrary level with associated Galois representation V_f . Let F be a finite abelian extension of \mathbf{Q} with cyclotomic \mathbf{Z}_p -extension F_∞ . Recall that the p -adic Iwasawa main conjecture for f over F asserts that the Selmer group $\text{Sel}(F_\infty, A_f)$ is Λ -cotorsion and that the characteristic ideal of its dual is generated by the p -adic L -function $L_p(F_\infty, f)$. In fact, when the residual representation of V_f is absolutely irreducible, it is known by work of Kato that $\text{Sel}(F_\infty, A_f)$ is indeed Λ -cotorsion and that $L_p(F_\infty, f)$ is an element of the characteristic ideal of $\text{Sel}(F_\infty, A_f)$. In particular, this reduces the verification of the main conjecture for f over F to the equality of the algebraic and analytic Iwasawa invariants of f over F . The identical transition formulae in Theorems 2.8 and 3.1 thus yield the following immediate application to the main conjecture.

Theorem 4.2. *Let F'/F be a finite p -extension with F' abelian over \mathbf{Q} . If the residual representation of V_f is absolutely irreducible and p -distinguished, then the main conjecture holds for f over F with $\mu(F_\infty, f) = 0$ if and only if it holds for f over F' with $\mu(F'_\infty, f) = 0$.*

For an example of Theorem 4.2, consider the eigenform

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24}$$

of weight 12 and level 1. We take $p = 11$. It is well known that Δ is congruent modulo 11 to the newform associated to the elliptic curve $X_0(11)$. The 11-adic main conjecture is known for $X_0(11)$ over \mathbf{Q} ; it has trivial μ -invariant and λ -invariant equal to 1 (see, for instance, [1, Example 5.3.1]. We should be clear here that the non-triviality of λ in this case corresponds to a trivial zero of the p -adic L -function; we are using the Greenberg Selmer group which does account for the trivial zero.) It follows from [1] that the 11-adic main conjecture also holds for Δ over \mathbf{Q} , again with trivial μ -invariant and λ -invariant equal to 1. Theorem 4.2 thus allows us to conclude that the main conjecture holds for Δ over any abelian 11-extension of \mathbf{Q} .

For a specific example, consider $F = \mathbf{Q}(\zeta_{23})^+$; it is a cyclic 11-extension of \mathbf{Q} . We can easily use Theorem 4.1 to compute its λ -invariant: using that $\tau(23) = 18643272$ one finds that $h_{23}(\Delta) = 0$, so that $\lambda(\mathbf{Q}(\zeta_{23})^+, \Delta) = 11$.

For a more interesting example, take F to be the unique subfield of $\mathbf{Q}(\zeta_{1123})$ which is cyclic of order 11 over \mathbf{Q} . In this case we have

$$\tau(1123) \equiv 2 \pmod{11}$$

so that we have $h_{1123}(\Delta) = 20$. Thus, in this case, Theorem 4.1 shows that $\lambda(F, \Delta) = 31$.

4.3. The supersingular case. As mentioned in the introduction, the underlying principle of this paper is that the existence of a formula relating the λ -invariants of congruent Galois representations should imply a Kida-type formula for these invariants. We illustrate this now in the case of modular forms of weight two that are supersingular at p .

Let f be an eigenform of weight 2 and level N with Fourier coefficients in K some finite extension of \mathbf{Q}_p . Assume further than $p \nmid N$ and that $a_p(f)$ is not a p -adic

unit. In [8], Perrin-Riou associates to f a pair of algebraic and analytic μ -invariants over \mathbf{Q}_∞ which we denote by $\mu_\pm^*(\mathbf{Q}_\infty, f)$. (Here \star denotes either “alg” or “an” for algebraic and analytic respectively.) Moreover, when $\mu_+^*(\mathbf{Q}_\infty, f) = \mu_-^*(\mathbf{Q}_\infty, f)$ or when $a_p(f) = 0$, she also defines corresponding λ -invariants $\lambda_\pm^*(\mathbf{Q}_\infty, f)$. When $a_p(f) = 0$ these invariants coincide with the Iwasawa invariants of [6] and [9]. We also note that in [8] only the case of elliptic curves is treated, but the methods used there generalize to weight two modular forms.

We extend the definition of these invariants to the cyclotomic \mathbf{Z}_p -extension of an unramified abelian extension F of \mathbf{Q} . We define

$$\mu_\pm^*(F_\infty, f) = \sum_{\psi \in \text{Gal}(F/\mathbf{Q})^\vee} \mu_\pm^*(\mathbf{Q}_\infty, f_\psi) \quad \text{and} \quad \lambda_\pm^*(F_\infty, f) = \sum_{\psi \in \text{Gal}(F/\mathbf{Q})^\vee} \lambda_\pm^*(\mathbf{Q}_\infty, f_\psi)$$

for $\star \in \{\text{alg}, \text{an}\}$.

The following transition formula follows from the congruence results of [2].

Theorem 4.3. *Let f be as above and consider a p -extension of number fields F'/F with F'/\mathbf{Q} unramified at p . If $\mu_\pm^*(F_\infty, f) = 0$, then $\mu_\pm^*(F'_\infty, f) = 0$. Moreover, if this is the case, then*

$$\lambda_\pm^*(F'_\infty, f) = [F'_\infty : F_\infty] \cdot \lambda_\pm^*(F_\infty, f) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty, w'}/F_{\infty, w}, V_f).$$

In particular, if the main conjecture is true for f over F (with $\mu_\pm^(F_\infty, f) = 0$), then the main conjecture is true for f over F' (with $\mu_\pm^*(F'_\infty, f) = 0$).*

Proof. The proof of this theorem proceeds along the lines of the proof of Theorem 3.1 replacing the appeals to the results of [1, 11] to the results of [2]. The main result of [2] is a formula relating the λ_\pm^* -invariants of congruent supersingular weight two modular forms. This formula has the same shape as the formulas that appear in [1] and [11] which allows for the proof to proceed nearly verbatim. \square

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