

KIDA'S FORMULA AND CONGRUENCES

ROBERT POLLACK AND TOM WESTON

1. INTRODUCTION

Let f be a modular eigenform of weight at least two and let F be a finite abelian extension of \mathbf{Q} . Fix an odd prime p at which f is ordinary in the sense that the p^{th} Fourier coefficient of f is not divisible by p . In Iwasawa theory, one associates two objects to f over the cyclotomic \mathbf{Z}_p -extension F_∞ of F : a Selmer group $\text{Sel}(F_\infty, A_f)$ (where A_f denotes the divisible version of the two-dimensional Galois representation attached to f) and a p -adic L -function $L_p(F_\infty, f)$. In this paper we prove a formula, generalizing work of Kida and Hachimori–Matsuno, relating the Iwasawa invariants of these objects over F with their Iwasawa invariants over p -extensions of F .

For Selmer groups our results are significantly more general. Let T be a lattice in a nearly ordinary p -adic Galois representation V ; set $A = V/T$. When $\text{Sel}(F_\infty, A)$ is a cotorsion Iwasawa module, its Iwasawa μ -invariant $\mu^{\text{alg}}(F_\infty, A)$ is said to vanish if $\text{Sel}(F_\infty, A)$ is cofinitely generated and its λ -invariant $\lambda^{\text{alg}}(F_\infty, A)$ is simply its p -adic corank. We prove the following result relating these invariants in a p -extension.

Theorem 1. *Let F'/F be a finite Galois p -extension that is unramified at all places dividing p . Assume that T satisfies the technical assumptions (1)–(5) of Section 2. If $\text{Sel}(F_\infty, A)$ is Λ -cotorsion with $\mu^{\text{alg}}(F_\infty, A) = 0$, then $\text{Sel}(F'_\infty, A)$ is Λ -cotorsion with $\mu^{\text{alg}}(F'_\infty, A) = 0$. Moreover, in this case*

$$\lambda^{\text{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\text{alg}}(F_\infty, A) + \sum_{w'} m(F'_{\infty, w'} / F_{\infty, w}, V)$$

where the sum extends over places w' of F'_∞ which are ramified in F'_∞ / F_∞ .

If V is associated to a cuspform f and F' is an abelian extension of \mathbf{Q} , then the same results hold for the analytic Iwasawa invariants of f .

Here $m(F'_{\infty, w'} / F_{\infty, w}, V)$ is a certain difference of local multiplicities defined in Section 2.1. In the case of Galois representations associated to Hilbert modular forms, these local factors can be made quite explicit; see Section 4.1 for details.

It follows from Theorem 1 and work of Kato that if the p -adic main conjecture holds for a modular form f over \mathbf{Q} , then it holds for f over all abelian p -extensions of \mathbf{Q} ; see Section 4.2 for details.

These Riemann–Hurwitz type formulas were first discovered by Kida [5] in the context of λ -invariants of CM fields. More precisely, when F'/F is a p -extension of CM fields and $\mu^-(F_\infty/F) = 0$, Kida gave a precise formula for $\lambda^-(F'_\infty/F')$ in terms of $\lambda^-(F_\infty/F)$ and local data involving the primes that ramify in F'/F . (See also [4] for a representation theoretic interpretation of Kida's result.) A similar formula in a somewhat different setting was given for elliptic curves with complex multiplication at ordinary primes by Wingberg [12]; Hachimori–Matsuno [3] established

Supported by NSF grants DMS-0439264 and DMS-0440708.

the cyclotomic version in general. The analytic analogue was first established for ideal class groups by Sinnott [10] and for elliptic curves by Matsuno [7].

Our proof is most closely related to the arguments in [10] and [7] where congruences implicitly played a large role in their study of analytic λ -invariants. In this paper, we make the role of congruences more explicit and apply these methods to study both algebraic and analytic λ -invariants.

As is usual, we first reduce to the case where F'/F is abelian. (Some care is required to show that our local factors are well behaved in towers of fields; this is discussed in Section 2.1.) In this case, the λ -invariant of V over F' can be expressed as the sum of the λ -invariants of twists of V by characters of $\text{Gal}(F'/F)$. The key observation (already visible in both [10] and [7]) is that since $\text{Gal}(F'/F)$ is a p -group, all of its characters are trivial modulo a prime over p and, thus, the twisted Galois representations are all congruent to V modulo a prime over p . The algebraic case of Theorem 1 then follows from the results of [11] which gives a precise local formula for the difference between λ -invariants of congruent Galois representations. The analytic case is handled similarly using the results of [1].

The basic principle behind this argument is that a formula relating the Iwasawa invariants of congruent Galois representations should imply of a transition formula for these invariants in p -extensions. As an example of this, in Section 4.3, we use results of [2] to prove a Kida formula for the Iwasawa invariants (in the sense of [8, 6, 9]) of weight 2 modular forms at supersingular primes.

Acknowledgments: We would like to thank the anonymous referee for several helpful comments and for pointing out some errors in an earlier draft of this paper.

2. ALGEBRAIC INVARIANTS

2.1. Local preliminaries. We begin by studying the local terms that appear in our results. Fix distinct primes ℓ and p and let L denote a finite extension of the cyclotomic \mathbf{Z}_p -extension of \mathbf{Q}_ℓ . Fix a field K of characteristic zero and a finite-dimensional K -vector space V endowed with a continuous K -linear action of the absolute Galois group G_L of L . Set

$$m_L(V) := \dim_K (V_{I_L})^{G_L},$$

the multiplicity of the trivial representation in the I_L -coinvariants of V . Note that this multiplicity is invariant under extension of scalars, so that we can enlarge K as necessary.

Let L' be a finite Galois p -extension of L . Note that L' must be cyclic and totally ramified since L contains the \mathbf{Z}_p -extension of \mathbf{Q}_ℓ . Let G denote the Galois group of L'/L . Assuming that K contains all $[L' : L]^{\text{th}}$ roots of unity, for a character $\chi : G \rightarrow K^\times$ of G , we set $V_\chi = V \otimes_K K(\chi)$ with $K(\chi)$ a one-dimensional K -vector space on which G acts via χ . We define

$$m(L'/L, V) := \sum_{\chi \in G^\vee} m_L(V) - m_L(V_\chi)$$

where G^\vee denotes the K -dual of G .

The next result shows how these invariants behave in towers of fields.

Lemma 2.1. *Let L'' be a finite Galois p -extension of L and let L' be a Galois extension of L contained in L'' . Assume that K contains all $[L'' : L]^{\text{th}}$ roots of*

unity. Then

$$m(L''/L, V) = [L'' : L'] \cdot m(L'/L, V) + m(L''/L', V).$$

Proof. Set $G = \text{Gal}(L''/L)$ and $H = \text{Gal}(L''/L')$. Consider the Galois group $G_L/I_{L''}$ over L of the maximal unramified extension of L'' . It sits in an exact sequence

$$(1) \quad 0 \rightarrow G_{L''}/I_{L''} \rightarrow G_L/I_{L''} \rightarrow G \rightarrow 0$$

which is in fact split since the maximal unramified extensions of both L and L'' are obtained by adjoining all prime-to- p roots of unity.

Fix a character $\chi \in G^\vee$. We compute

$$\begin{aligned} m_L(V_\chi) &= \dim_K((V_\chi)_{I_L})^{G_L} \\ &= \dim_K\left(\left(\left(\left(V_\chi\right)_{I_{L''}}\right)_G\right)^{G_{L''}}\right)^G \\ &= \dim_K\left(\left(\left(\left(V_\chi\right)_{I_{L''}}\right)^{G_{L''}}\right)_G\right)^G \quad \text{since (1) is split} \\ &= \dim_K\left(\left(\left(V_\chi\right)_{I_{L''}}\right)^{G_{L''}}\right)^G \quad \text{since } G \text{ is finite cyclic} \\ &= \dim_K\left(\left(V_{I_{L''}}\right)^{G_{L''}} \otimes \chi\right)^G \quad \text{since } \chi \text{ is trivial on } G_{L''}. \end{aligned}$$

The lemma thus follows from the following purely group-theoretical statement applied with $W = (V_{I_{L''}})^{G_{L''}}$: for a finite dimensional representation W of a finite abelian group G over a field of characteristic zero containing $\mu_{\#G}$, we have

$$\begin{aligned} \sum_{\chi \in G^\vee} (\langle W, 1 \rangle_G - \langle W, \chi \rangle_G) &= \\ \#H \cdot \sum_{\chi \in (G/H)^\vee} (\langle W, 1 \rangle_G - \langle W, \chi \rangle_G) &+ \sum_{\chi \in H^\vee} (\langle W, 1 \rangle_H - \langle W, \chi \rangle_H) \end{aligned}$$

for any subgroup H of G ; here $\langle W, \chi \rangle_G$ (resp. $\langle W, \chi \rangle_H$) is the multiplicity of the character χ in W regarded as a representation of G (resp. H). To prove this, we compute

$$\begin{aligned} &\sum_{\chi \in G^\vee} (\langle W, 1 \rangle_G - \langle W, \chi \rangle_G) \\ &= \#G \cdot \langle W, 1 \rangle_G - \left\langle W, \text{Ind}_1^G 1 \right\rangle_G \\ &= \#G \cdot \langle W, 1 \rangle_G - \#H \cdot \left\langle W, \text{Ind}_H^G 1 \right\rangle_G + \#H \cdot \left\langle W, \text{Ind}_H^G 1 \right\rangle_G - \left\langle W, \text{Ind}_1^G 1 \right\rangle_G \\ &= \#H \cdot \sum_{\chi \in (G/H)^\vee} (\langle W, 1 \rangle_G - \langle W, \chi \rangle_G) + \sum_{\chi \in H^\vee} \left(\left\langle W, \text{Ind}_H^G 1 \right\rangle_G - \left\langle W, \text{Ind}_H^G \chi \right\rangle_G \right) \\ &= \#H \cdot \sum_{\chi \in (G/H)^\vee} (\langle W, 1 \rangle_G - \langle W, \chi \rangle_G) + \sum_{\chi \in H^\vee} (\langle W, 1 \rangle_H - \langle W, \chi \rangle_H) \end{aligned}$$

by Frobenius reciprocity. \square

2.2. Global preliminaries. Fix a number field F ; for simplicity we assume that F is either totally real or totally imaginary. Fix also an odd prime p and a finite extension K of \mathbf{Q}_p ; we write \mathcal{O} for the ring of integers of K , π for a fixed choice of uniformizer of \mathcal{O} , and $k = \mathcal{O}/\pi$ for the residue field of \mathcal{O} .

Let T be a nearly ordinary Galois representation over F with coefficients in \mathcal{O} ; that is, T is a free \mathcal{O} -module of some rank n endowed with an \mathcal{O} -linear action of the absolute Galois group G_F , together with a choice for each place v of F dividing p of a complete flag

$$0 = T_v^0 \subset T_v^1 \subset \cdots \subset T_v^n = T$$

stable under the action of the decomposition group $G_v \subseteq G_F$ of v . We make the following assumptions on T :

- (1) For each place v dividing p we have

$$(T_v^i/T_v^{i-1}) \otimes k \not\cong (T_v^j/T_v^{j-1}) \otimes k$$

as $k[G_v]$ -modules for all $i \neq j$;

- (2) If F is totally real, then $\text{rank } T^{c_v=1}$ is independent of the archimedean place v (here c_v is a complex conjugation at v);
(3) If F is totally imaginary, then n is even.

Remark 2.2. The conditions above are significantly more restrictive than are actually required to apply the results of [11]. As our main interest is in abelian (and thus necessarily Galois) extensions of \mathbf{Q} , we have chosen to include the assumptions (2) and (3) to simplify the exposition. The assumption (1) is also stronger than necessary: all that is actually needed is that the centralizer of $T \otimes k$ consists entirely of scalars and that $\mathfrak{gl}_n/\mathfrak{b}_v$ has trivial adjoint G_v -invariants for all places v dividing p ; here \mathfrak{gl}_n denotes the p -adic Lie algebra of GL_n and \mathfrak{b}_v denotes the p -adic Lie algebra of the Borel subgroup associated to the complete flag at v . In particular, when T has rank 2, we may still allow the case that $T \otimes k$ has the form

$$\begin{pmatrix} \chi & * \\ 0 & \chi \end{pmatrix}$$

so long as $*$ is non-trivial. (Equivalently, if T is associated to a modular form f , the required assumption is that f is p -distinguished.)

Set $A = T \otimes_{\mathcal{O}} K/\mathcal{O}$; it is a cofree \mathcal{O} -module of corank n with an \mathcal{O} -linear action of G_F . Let c equal the rank of $T^{c_v=1}$ (resp. $n/2$) if F is totally real (resp. totally imaginary) and set

$$A_v^{\text{cr}} := \text{im}(T_v^c \otimes_{\mathcal{O}} K \hookrightarrow T \otimes_{\mathcal{O}} K \rightarrow A).$$

We define the Selmer group of A over the cyclotomic \mathbf{Z}_p -extension F_{∞} of F by

$$\text{Sel}(F_{\infty}, A) = \ker \left(H^1(F_{\infty}, A) \rightarrow \left(\bigoplus_{w \nmid p} H^1(F_{\infty, w}, A) \right) \times \left(\bigoplus_{w \mid p} H^1(F_{\infty, w}, A/A_v^{\text{cr}}) \right) \right).$$

The Selmer group $\text{Sel}(F_{\infty}, A)$ is naturally a module for the Iwasawa algebra $\Lambda_{\mathcal{O}} := \mathcal{O}[[\text{Gal}(F_{\infty}/F)]]$. If $\text{Sel}(F_{\infty}, A)$ is $\Lambda_{\mathcal{O}}$ -cotorsion (that is, if the dual of $\text{Sel}(F_{\infty}, A)$ is a torsion $\Lambda_{\mathcal{O}}$ -module), then we write $\mu^{\text{alg}}(F_{\infty}, A)$ and $\lambda^{\text{alg}}(F_{\infty}, A)$ for its Iwasawa invariants; in particular, $\mu^{\text{alg}}(F_{\infty}, A) = 0$ if and only if $\text{Sel}(F_{\infty}, A)$ is a cofinitely generated \mathcal{O} -module, while $\lambda^{\text{alg}}(F_{\infty}, A)$ is the \mathcal{O} -corank of $\text{Sel}(F_{\infty}, A)$.

Remark 2.3. In the case that T is in fact an *ordinary* Galois representation (meaning that the action of inertia on each T_v^i/T_v^{i-1} is by an integer power e_i (independent of v) of the cyclotomic character such that $e_1 > e_2 > \dots > e_n$), then our Selmer group $\text{Sel}(F_\infty, A)$ is simply the Selmer group in the sense of Greenberg of a twist of A ; see [11, Section 1.3] for details.

2.3. Extensions. Let F' be a finite Galois extension of F with degree equal to a power of p . We write F'_∞ for the cyclotomic \mathbf{Z}_p -extension of F' and set $G = \text{Gal}(F'_\infty/F_\infty)$. Note that T satisfies hypotheses (1)–(3) over F' as well, so that we may define $\text{Sel}(F'_\infty, A)$ analogously to $\text{Sel}(F_\infty, A)$. (For (1) this follows from the fact that G_v acts on $(T_v^i/T_v^{i-1}) \otimes k$ by a character of prime-to- p order; for (2) and (3) it follows from the fact that p is assumed to be odd.)

Lemma 2.4. *The restriction map*

$$(2) \quad \text{Sel}(F_\infty, A) \rightarrow \text{Sel}(F'_\infty, A)^G$$

has finite kernel and cokernel.

Proof. This is straightforward from the definitions and the fact that G is finite and A is cofinitely generated; see [3, Lemma 3.3] for details. \square

We can use Lemma 2.4 to relate the μ -invariants of A over F_∞ and F'_∞ .

Corollary 2.5. *If $\text{Sel}(F_\infty, A)$ is Λ -cotorsion with $\mu^{\text{alg}}(F_\infty, A) = 0$, then $\text{Sel}(F'_\infty, A)$ is Λ -cotorsion with $\mu^{\text{alg}}(F'_\infty, A) = 0$.*

Proof. This is a straightforward argument using Lemma 2.4 and Nakayama's lemma for compact local rings; see [3, Corollary 3.4] for details. \square

Fix a finite extension K' of K containing all $[F' : F]^{\text{th}}$ roots of unity. Consider a character $\chi : G \rightarrow \mathcal{O}'^\times$ taking values in the ring of integers \mathcal{O}' of K' ; note that χ is necessarily even since $[F' : F]$ is odd. We set

$$A_\chi = A \otimes_{\mathcal{O}} \mathcal{O}'(\chi)$$

where $\mathcal{O}'(\chi)$ is a free \mathcal{O}' -module of rank one with G_{F_∞} -action given by χ . If we give A_χ the induced complete flags at places dividing p , then A_χ satisfies hypotheses (1)–(3) and we have

$$A_{\chi, v}^{\text{cr}} = A_v^{\text{cr}} \otimes_{\mathcal{O}} \mathcal{O}'(\chi) \subseteq A_\chi$$

for each place v dividing p . We write $\text{Sel}(F_\infty, A_\chi)$ for the corresponding Selmer group, regarded as a $\Lambda_{\mathcal{O}'}$ -module; in particular, by $\lambda^{\text{alg}}(F_\infty, A_\chi)$ we mean the \mathcal{O}' -corank of $\text{Sel}(F_\infty, A_\chi)$, rather than the \mathcal{O} -corank. We write G^\vee for the set of all characters $\chi : G \rightarrow \mathcal{O}'^\times$.

Proposition 2.6. *Assume that $\text{Sel}(F_\infty, A)$ is Λ -cotorsion with $\mu^{\text{alg}}(F_\infty, A) = 0$. If G is an abelian group, then there is a natural map*

$$\bigoplus_{\chi \in G^\vee} \text{Sel}(F_\infty, A_\chi) \rightarrow \text{Sel}(F'_\infty, A) \otimes_{\mathcal{O}} \mathcal{O}'$$

with finite kernel and cokernel.

Proof. First note that as $\mathcal{O}'[[G_{F'}]]$ -modules we have

$$A \otimes_{\mathcal{O}} \mathcal{O}' \cong A_\chi$$

from which it easily follows that

$$(3) \quad (\mathrm{Sel}(F'_\infty, A) \otimes_{\mathcal{O}} \mathcal{O}'(\chi))^G = \mathrm{Sel}(F'_\infty, A_\chi)^G.$$

Also, for any cofinitely generated $\mathcal{O}[G]$ -module S , the natural map

$$(4) \quad \bigoplus_{\chi \in G^\vee} (S \otimes \mathcal{O}'(\chi))^G \rightarrow S \otimes \mathcal{O}'$$

has finite kernel and cokernel. Since we are assuming that $\mu^{\mathrm{alg}}(F_\infty, A) = 0$, we may take $S = \mathrm{Sel}(F'_\infty, A)$ in (4); combining this with (3) yields a map

$$\bigoplus_{\chi \in G^\vee} (\mathrm{Sel}(F'_\infty, A_\chi))^G \rightarrow \mathrm{Sel}(F'_\infty, A_\chi) \otimes \mathcal{O}'$$

with finite kernel and cokernel. Now applying Lemma 2.4 for each twist A_χ , we obtain our proposition. \square

As an immediate corollary, we have the following.

Corollary 2.7. *If $\mathrm{Sel}(F_\infty, A)$ is Λ -cotorsion with $\mu^{\mathrm{alg}}(F_\infty, A) = 0$, then each group $\mathrm{Sel}(F_\infty, A_\chi)$ is $\Lambda_{\mathcal{O}'}$ -cotorsion with $\mu^{\mathrm{alg}}(F_\infty, A_\chi) = 0$. Moreover, if G is abelian, then*

$$\lambda^{\mathrm{alg}}(F'_\infty, A) = \sum_{\chi \in G^\vee} \lambda^{\mathrm{alg}}(F_\infty, A_\chi).$$

2.4. Algebraic transition formula. We continue with the notation of the previous section. We write $R(F'_\infty/F_\infty)$ for the set of prime-to- p places of F'_∞ which are ramified in F'_∞/F_∞ . For a place $w' \in R(F'_\infty/F_\infty)$, we write w for its restriction to F_∞ .

Theorem 2.8. *Let F'/F be a finite Galois p -extension with Galois group G which is unramified at all places dividing p . Let T be a nearly ordinary Galois representation over F with coefficients in \mathcal{O} satisfying (1)–(3). Set $A = T \otimes K/\mathcal{O}$ and assume that:*

- (4) $H^0(F, A[\pi]) = H^0(F, \mathrm{Hom}(A[\pi], \mu_p)) = 0$;
- (5) $H^0(I_v, A/A_v^{cr})$ is \mathcal{O} -divisible for all v dividing p .

If $\mathrm{Sel}(F_\infty, A)$ is Λ -cotorsion with $\mu^{\mathrm{alg}}(F_\infty, A) = 0$, then $\mathrm{Sel}(F'_\infty, A)$ is Λ -cotorsion with $\mu^{\mathrm{alg}}(F'_\infty, A) = 0$. Moreover, in this case,

$$\lambda^{\mathrm{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\mathrm{alg}}(F_\infty, A) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty, w'}/F_{\infty, w}, V)$$

with $V = T \otimes K$ and $m(F'_{\infty, w'}/F_{\infty, w}, V)$ as in Section 2.1.

Note that $m(F'_{\infty, w'}/F_{\infty, w}, V)$ in fact depends only on w and not on w' . The hypotheses (4) and (5) are needed to apply the results of [11]; they will not otherwise appear in the proof below. We note that the assumption that F'/F is unramified at p is primarily needed to assure that the condition (5) holds for twists of A as well.

Since p -groups are solvable and the only simple p -group is cyclic, the next lemma shows that it suffices to consider the case of $\mathbf{Z}/p\mathbf{Z}$ -extensions.

Lemma 2.9. *Let F''/F be a Galois p -extension of number fields and let F' be an intermediate extension which is Galois over F . Let T be as above. If Theorem 2.8 holds for T with respect to any two of the three field extensions F''/F' , F'/F and F''/F , then it holds for T with respect to the third extension.*

Proof. This is clear from Corollary 2.5 except for the λ -invariant formula. Substituting the formula for $\lambda(F'_\infty, A)$ in terms of $\lambda(F_\infty, A)$ into the formula for $\lambda(F''_\infty, A)$ in terms of $\lambda(F'_\infty, A)$, one finds that it suffices to show that

$$\begin{aligned} \sum_{w'' \in R(F''_\infty/F_\infty)} m(F''_{\infty, w''}/F_{\infty, w}, V) = \\ [F''_\infty : F'_\infty] \cdot \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty, w'}/F_{\infty, w}, V) \\ + \sum_{w'' \in R(F''_\infty/F'_\infty)} m(F''_{\infty, w''}/F'_{\infty, w'}, V). \end{aligned}$$

This formula follows upon summing the formula of Lemma 2.1 over all $w'' \in R(F''_\infty/F_\infty)$ and using the two facts:

- $[F''_\infty : F'_\infty]/[F''_{\infty, w''} : F'_{\infty, w'}]$ equals the number of places of F''_∞ lying over w' (since the residue field of $F_{\infty, w}$ has no p -extensions);
- $m(F''_{\infty, w''}/F'_{\infty, w'}, V) = 0$ for any $w'' \in R(F''_\infty/F_\infty) - R(F''_\infty/F'_\infty)$.

□

Proof of Theorem 2.8. By Lemma 2.9 and the preceding remark, we may assume that F'_∞/F_∞ is a cyclic extension of degree p . The fact that $\text{Sel}(F'_\infty, A)$ is cotorsion with trivial μ -invariant is simply Corollary 2.5. Furthermore, by Corollary 2.7, we have

$$\lambda^{\text{alg}}(F'_\infty, A) = \sum_{\chi \in G^\vee} \lambda^{\text{alg}}(F_\infty, A_\chi).$$

For $\chi \in G^\vee$, note that χ is trivial modulo a uniformizer π' of \mathcal{O}' as it takes values in μ_p . In particular, the residual representations $A_\chi[\pi']$ and $A[\pi]$ are isomorphic. Under the hypotheses (1)–(5), the result [11, Theorem 1] gives a precise formula for the relation between λ -invariants of congruent Galois representations. In the present case it takes the form:

$$\lambda^{\text{alg}}(F_\infty, A_\chi) = \lambda^{\text{alg}}(F_\infty, A) + \sum_{w' \nmid p} (m_{F_{\infty, w}}(V \otimes \omega^{-1}) - m_{F_{\infty, w}}(V_\chi \otimes \omega^{-1}))$$

where the sum is over all prime-to- p places w' of F'_∞ , w denotes the place of F_∞ lying under w' and ω is the mod p cyclotomic character. The only non-zero terms in this sum are those for which w' is ramified in F'_∞/F_∞ . For any such w' , we have $\mu_p \subseteq F_{\infty, w}$ by local class field theory so that ω is in fact trivial at w ; thus

$$\lambda^{\text{alg}}(F_\infty, A_\chi) = \lambda^{\text{alg}}(F_\infty, A) + \sum_{w' \in R(F'_\infty/F_\infty)} (m_{F_{\infty, w}}(V) - m_{F_{\infty, w}}(V_\chi)).$$

Summing over all $\chi \in G^\vee$ then yields

$$\lambda^{\text{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\text{alg}}(F_\infty, A) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty, w'}/F_{\infty, w}, V)$$

which completes the proof. □

3. ANALYTIC INVARIANTS

3.1. Definitions. Let $f = \sum a_n q^n$ be a modular eigenform of weight $k \geq 2$, level N and character ε . Let K denote the finite extension of \mathbf{Q}_p generated by the Fourier coefficients of f (under some fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$), let \mathcal{O} denote the ring of integers of K and let k denote the residue field of \mathcal{O} . Let V_f denote a two-dimensional K -vector space with Galois action associated to f in the usual way; thus the characteristic polynomial of a Frobenius element at a prime $\ell \nmid Np$ is

$$x^2 - a_\ell x + \ell^{k-1} \varepsilon(\ell).$$

Fix a Galois stable \mathcal{O} -lattice T_f in V_f . We assume that $T_f \otimes k$ is an irreducible Galois representation; in this case T_f is uniquely determined up to scaling. Set $A_f = T_f \otimes K/\mathcal{O}$.

Assuming that f is p -ordinary (in the sense that a_p is relatively prime to p) and fixing a canonical period for f , one can associate to f a p -adic L -function $L_p(\mathbf{Q}_\infty/\mathbf{Q}, f)$ which lies in $\Lambda_{\mathcal{O}}$. This is well-defined up to a p -adic unit (depending upon the choice of a canonical period) and thus has well-defined Iwasawa invariants.

Let F/\mathbf{Q} be a finite abelian extension and let F_∞ denote the cyclotomic \mathbf{Z}_p -extension of F . For a character χ of $\text{Gal}(F/\mathbf{Q})$, we denote by f_χ the modular eigenform $\sum a_n \chi(n) q^n$ obtained from f by twisting by χ (viewed as a Dirichlet character). If f is p -ordinary and F/\mathbf{Q} is unramified at p , then f_χ is again p -ordinary and we define

$$L_p(F_\infty/F, f) = \prod_{\chi \in \text{Gal}(F/\mathbf{Q})^\vee} L_p(\mathbf{Q}_\infty/\mathbf{Q}, f_\chi).$$

If F/\mathbf{Q} is ramified at p , it is still possible to define $L_p(F_\infty/F, f)$; see [7, pg. 5], for example.

If F_1 and F_2 are two distinct number fields whose cyclotomic \mathbf{Z}_p -extensions agree, the corresponding p -adic L -functions of f over F_1 and F_2 need not agree. However, it is easy to check that the λ -invariants of these two power series are equal while their μ -invariants differ by a factor of a power of p . As we are only interested in the case of vanishing μ -invariants, we will abuse notation somewhat and simply denote the Iwasawa invariants of $L_p(F_\infty/F, f)$ by $\mu^{\text{an}}(F_\infty, f)$ and $\lambda^{\text{an}}(F_\infty, f)$.

3.2. Analytic transition formula. Let F/\mathbf{Q} be a finite abelian extension of \mathbf{Q} and let F' be a finite p -extension of F such that F'/\mathbf{Q} is abelian. As always, let F_∞ and F'_∞ denote the cyclotomic \mathbf{Z}_p -extensions of F and F' . As before, we write $R(F'_\infty/F_\infty)$ for the set of prime-to- p places of F'_∞ which are ramified in F'_∞/F_∞ .

Theorem 3.1. *Let f be a p -ordinary modular form such that $T_f \otimes k$ is irreducible and p -distinguished. If $\mu^{\text{an}}(F_\infty, f) = 0$, then $\mu^{\text{an}}(F'_\infty, f) = 0$. Moreover, if this is the case, then*

$$\lambda^{\text{an}}(F'_\infty, f) = [F'_\infty : F_\infty] \cdot \lambda^{\text{an}}(F_\infty, f) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty, w'}/F_{\infty, w}, V_f).$$

Proof. By Lemma 2.9, we may assume $[F : \mathbf{Q}]$ is prime-to- p . Indeed, let F_0 be the maximal subfield of F of prime-to- p degree over \mathbf{Q} . By Lemma 2.9, knowledge of the theorem for the two extensions F'/F_0 and F/F_0 would then imply it for F'/F as well. Furthermore, replacing F (resp. F') by the maximal tamely ramified subextension of F_∞ (resp. F'_∞), we may assume that every character of $\text{Gal}(F/\mathbf{Q})$

and $\text{Gal}(F'/\mathbf{Q})$ is the product of a power of the mod p cyclotomic character and a character unramified at p .

After making these reductions, we let M denote the (unique) p -extension of \mathbf{Q} inside of F' such that $MF = F'$. Set $G = \text{Gal}(F/\mathbf{Q})$ and $H = \text{Gal}(M/\mathbf{Q})$, so that $\text{Gal}(F'/\mathbf{Q}) \cong G \times H$. We have

$$(5) \quad \mu^{\text{an}}(F_\infty, f) = \sum_{\psi \in \text{Gal}(F/\mathbf{Q})^\vee} \mu^{\text{an}}(\mathbf{Q}_\infty, f_\psi)$$

and

$$(6) \quad \mu^{\text{an}}(F'_\infty, f) = \sum_{\psi \in \text{Gal}(F'/\mathbf{Q})^\vee} \mu^{\text{an}}(\mathbf{Q}_\infty, f_\psi) = \sum_{\psi \in G^\vee} \sum_{\chi \in H^\vee} \mu^{\text{an}}(\mathbf{Q}_\infty, f_{\psi\chi}).$$

Since we are assuming that $\mu^{\text{an}}(F_\infty, f) = 0$ and since these μ -invariants are non-negative, from (5) it follows that $\mu^{\text{an}}(\mathbf{Q}_\infty, f_\psi) = 0$ for each $\psi \in \text{Gal}(F/\mathbf{Q})^\vee$.

Fix $\psi \in G^\vee$. For any $\chi \in H^\vee$, $\psi\chi$ is congruent to ψ modulo any prime over p and thus f_χ and $f_{\psi\chi}$ are congruent modulo any prime over p . Then, since $\mu^{\text{an}}(\mathbf{Q}_\infty, f_\psi) = 0$, by [1, Theorem 3.7.5] it follows that $\mu^{\text{an}}(\mathbf{Q}_\infty, f_{\psi\chi}) = 0$ for each $\chi \in H^\vee$. (Note that the results of [1] apply to twists of p -ordinary forms by powers of the mod p cyclotomic character; this is why the reduction to the tamely ramified case is necessary for this argument.) Therefore, by (6) we have that $\mu^{\text{an}}(F'_\infty, f) = 0$ proving the first part of the theorem.

For λ -invariants, we again have

$$\lambda^{\text{an}}(F_\infty, f) = \sum_{\psi \in \text{Gal}(F/\mathbf{Q})^\vee} \lambda^{\text{an}}(\mathbf{Q}_\infty, f_\psi).$$

and

$$(7) \quad \lambda^{\text{an}}(F'_\infty, f) = \sum_{\psi \in G^\vee} \sum_{\chi \in H^\vee} \lambda^{\text{an}}(\mathbf{Q}_\infty, f_{\psi\chi}).$$

By [1, Theorem 3.7.7] the congruence between f_χ and $f_{\psi\chi}$ implies that

$$\lambda^{\text{an}}(\mathbf{Q}_\infty, f_{\psi\chi}) - \lambda^{\text{an}}(\mathbf{Q}_\infty, f_\psi) = \sum_v (m_{\mathbf{Q}_{\infty,v}}(V_{f_{\psi\chi}} \otimes \omega^{-1}) - m_{\mathbf{Q}_{\infty,v}}(V_{f_\psi} \otimes \omega^{-1}))$$

where the sum is over all places v of \mathbf{Q}_∞ at which χ is ramified. (Note that in [1] the sum extends over all prime-to- p places; however, the terms are trivial unless χ is ramified at v . Also note that the mod p cyclotomic characters that appear are actually trivial since if $\mathbf{Q}_{\infty,v}$ has a ramified Galois p -extensions for $v \nmid p$, then $\mu_p \subseteq \mathbf{Q}_{\infty,v}$.)

Combining this with (7) and the definition of $m(M_{\infty, v'} / \mathbf{Q}_{\infty, v}, V_{f_{\psi}})$, we conclude that

$$\begin{aligned} \lambda^{\text{an}}(F'_{\infty}, f) &= \sum_{\psi \in G^{\vee}} \left([F'_{\infty} : F_{\infty}] \cdot \lambda^{\text{an}}(\mathbf{Q}_{\infty}, f_{\psi}) + \sum_{v' \in R(M_{\infty} / \mathbf{Q}_{\infty})} m(M_{\infty, v'} / \mathbf{Q}_{\infty, v}, V_{f_{\psi}}) \right) \\ &= [F'_{\infty} : F_{\infty}] \cdot \lambda^{\text{an}}(F_{\infty}, f) + \sum_{v' \in R(M_{\infty} / \mathbf{Q}_{\infty})} \sum_{\psi \in G^{\vee}} m(M_{\infty, v'} / \mathbf{Q}_{\infty, v}, V_{f_{\psi}}) \\ &= [F'_{\infty} : F_{\infty}] \cdot \lambda^{\text{an}}(F_{\infty}, f) + \sum_{v' \in R(M_{\infty} / \mathbf{Q}_{\infty})} g_{v'}(F'_{\infty} / M_{\infty}) \cdot \\ &\quad m(M_{\infty, v'} / \mathbf{Q}_{\infty, v}, \mathbf{Z}[\text{Gal}(F_{\infty, w} / \mathbf{Q}_{\infty, v})] \otimes V_f) \end{aligned}$$

where $g_{v'}(F'_{\infty} / M_{\infty})$ denotes the number of places of F'_{∞} above the place v' of M_{∞} . By Frobenius reciprocity,

$$m(M_{\infty, v'} / \mathbf{Q}_{\infty, v}, \mathbf{Z}[\text{Gal}(F_{\infty, w} / \mathbf{Q}_{\infty, v})] \otimes V_f) = m(F'_{\infty, w'} / F_{\infty, w}, V_f)$$

where w' is the unique place of F'_{∞} above v' and w . It follows that

$$\lambda(F'_{\infty}, f) = [F'_{\infty} : F_{\infty}] \cdot \lambda^{\text{an}}(F_{\infty}, f) + \sum_{w' \in R(F'_{\infty} / F_{\infty})} m(F'_{\infty, w'} / F_{\infty, w}, V_f)$$

as desired. \square

4. ADDITIONAL RESULTS

4.1. Hilbert modular forms. We illustrate our results in the case of the two-dimensional representation V_f associated to a Hilbert modular eigenform f over a totally real field F . Although in principle our analytic results should remain true in this context, we focus on the less conjectural algebraic picture. Fix a G_F -stable lattice $T_f \subseteq V_f$ and let $A_f = T_f \otimes K / \mathcal{O}$.

Let F' be a finite Galois p -extension of F unramified at all places dividing p ; for simplicity we assume also that F' is linearly disjoint from F_{∞} . Let v be a place of F not dividing p and fix a place v' of F' lying over v . For a character φ of G_v , we define

$$h(\varphi) = \begin{cases} -1 & \varphi \text{ ramified, } \varphi|_{G_{v'}}, \text{ unramified, and } \varphi \equiv 1 \pmod{\pi}; \\ 0 & \varphi \not\equiv 1 \pmod{\pi} \text{ or } \varphi|_{G_{v'}}, \text{ ramified}; \\ e_v(F'/F) - 1 & \varphi \text{ unramified and } \varphi \equiv 1 \pmod{\pi} \end{cases}$$

where $e_v(F'/F)$ denotes the ramification index of v in F'/F and $G_{v'}$ is the decomposition group at v' . Set

$$h_v(f) = \begin{cases} h(\varphi_1) + h(\varphi_2) & f \text{ principal series with characters } \varphi_1, \varphi_2 \text{ at } v; \\ h(\varphi) & f \text{ special with character } \varphi \text{ at } v; \\ 0 & f \text{ supercuspidal or extraordinary at } v. \end{cases}$$

For example, if f is unramified principal series at v with Frobenius characteristic polynomial

$$x^2 - a_v x + c_v,$$

then

$$h_v(f) = \begin{cases} 2(e_v(F'/F) - 1) & a_v \equiv 2, c_v \equiv 1 \pmod{\pi} \\ e_v(F'/F) - 1 & a_v \equiv c_v + 1 \not\equiv 2 \pmod{\pi} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.1. *Assume that f is ordinary (in the sense that for each place v dividing p the Galois representation V_f has a unique one-dimensional quotient unramified at v) and that*

$$H^0(F, A_f[\pi]) = H^0(F, \text{Hom}(A_f[\pi], \mu_p)) = 0.$$

If $\text{Sel}(F_\infty, A_f)$ is Λ -cotorsion with $\mu^{\text{alg}}(F_\infty, A_f) = 0$, then also $\text{Sel}(F'_\infty, A_f)$ is Λ -cotorsion with $\mu^{\text{alg}}(F'_\infty, A_f) = 0$ and

$$\lambda^{\text{alg}}(F'_\infty, A) = [F'_\infty : F_\infty] \cdot \lambda^{\text{alg}}(F_\infty, A) + \sum_v g_v(F'_\infty/F) \cdot h_v(f);$$

here the sum is over the prime-to- p places of F ramified in F'_∞ and $g_v(F'_\infty/F)$ denotes the number of places of F'_∞ lying over such a v .

Proof. Fix a place v of F not dividing p and let w denote a place of F_∞ lying over v . Since there are exactly $g_v(F_\infty/F)$ such places, by Theorem 2.8 it suffices to prove that

$$(8) \quad h_v(f) = m(F'_{\infty, w'}/F_{\infty, w}, V_f) := \sum_{\chi \in \text{Gal}(F'_{\infty, w'}/F_{\infty, w})^\vee} (m_{F_{\infty, w}}(V_f) - m_{F_{\infty, w}}(V_{f, \chi})).$$

This is a straightforward case analysis. We will discuss the case that V_f is special associated to a character φ at v ; the other cases are similar. In the special case, we have

$$V_{f, \chi}|_{I_{F_{\infty, w}}} = \begin{cases} K'(\chi\varphi) & \chi\varphi|_{G_{F_{\infty, w}}} \text{ unramified;} \\ 0 & \chi\varphi|_{G_{F_{\infty, w}}} \text{ ramified.} \end{cases}$$

Since an unramified character has trivial restriction to $G_{F_{\infty, w}}$ if and only if it has trivial reduction modulo π , it follows that

$$m_{F_{\infty, w}}(V_{f, \chi}) = \begin{cases} 1 & \varphi \equiv 1 \pmod{\pi} \text{ and } \chi\varphi|_{G_{F_{\infty, w}}} \text{ unramified;} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the sum in (8) is zero if $\varphi \not\equiv 1 \pmod{\pi}$ or if φ is ramified when restricted to $G_{F'_{\infty, w'}}$ (as then $\chi\varphi$ is ramified for all $\chi \in G_v^\vee$). If $\varphi \equiv 1 \pmod{\pi}$ and φ itself is unramified, then $m_{F_{\infty, w}}(V_f) = 1$ while $m_{F_{\infty, w}}(V_{f, \chi}) = 0$ for $\chi \neq 1$, so that the sum in (8) is $[F'_{\infty, w'} : F_{\infty, w}] - 1 = e_v(F'/F) - 1$, as desired. Finally, if $\varphi \equiv 1 \pmod{\pi}$ and φ is ramified but becomes unramified when restricted to $G_{v'}$, then $m_{F_{\infty, w}}(V_f) = 0$, while $m_{F_{\infty, w}}(V_{f, \chi}) = 1$ for a unique χ , so that the sum is -1 . \square

Suppose finally that f is in fact the Hilbert modular form associated to an elliptic curve E over F . The only principal series which occur are unramified and we have $c_v \equiv 1 \pmod{\pi}$ (since the determinant of V_f is cyclotomic and F_∞ has a p -extension (namely, F'_∞) ramified at v), so that

$$h_v(f) \neq 0 \Leftrightarrow a_v \equiv 2 \Leftrightarrow E(F_v) \text{ has a point of order } p$$

in which case $h_v(f) = 2(e_v(F'/F) - 1)$. The only characters which may occur in a special constituent are trivial or unramified quadratic, and we have $h_v(f) =$

$e_v(F'/F) - 1$ or 0 respectively. Thus Theorem 4.1 recovers [3, Theorem 3.1] in this case.

4.2. The main conjecture. Let f be a p -ordinary elliptic modular eigenform of weight at least two and arbitrary level with associated Galois representation V_f . Let F be a finite abelian extension of \mathbf{Q} with cyclotomic \mathbf{Z}_p -extension F_∞ . Recall that the p -adic Iwasawa main conjecture for f over F asserts that the Selmer group $\text{Sel}(F_\infty, A_f)$ is Λ -cotorsion and that the characteristic ideal of its dual is generated by the p -adic L -function $L_p(F_\infty, f)$. In fact, when the residual representation of V_f is absolutely irreducible, it is known by work of Kato that $\text{Sel}(F_\infty, A_f)$ is indeed Λ -cotorsion and that $L_p(F_\infty, f)$ is an element of the characteristic ideal of $\text{Sel}(F_\infty, A_f)$. In particular, this reduces the verification of the main conjecture for f over F to the equality of the algebraic and analytic Iwasawa invariants of f over F . The identical transition formulae in Theorems 2.8 and 3.1 thus yield the following immediate application to the main conjecture.

Theorem 4.2. *Let F'/F be a finite p -extension with F' abelian over \mathbf{Q} . If the residual representation of V_f is absolutely irreducible and p -distinguished, then the main conjecture holds for f over F with $\mu(F_\infty, f) = 0$ if and only if it holds for f over F' with $\mu(F'_\infty, f) = 0$.*

For an example of Theorem 4.2, consider the eigenform

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24}$$

of weight 12 and level 1. We take $p = 11$. It is well known that Δ is congruent modulo 11 to the newform associated to the elliptic curve $X_0(11)$. The 11-adic main conjecture is known for $X_0(11)$ over \mathbf{Q} ; it has trivial μ -invariant and λ -invariant equal to 1 (see, for instance, [1, Example 5.3.1]. We should be clear here that the non-triviality of λ in this case corresponds to a trivial zero of the p -adic L -function; we are using the Greenberg Selmer group which does account for the trivial zero.) It follows from [1] that the 11-adic main conjecture also holds for Δ over \mathbf{Q} , again with trivial μ -invariant and λ -invariant equal to 1. Theorem 4.2 thus allows us to conclude that the main conjecture holds for Δ over any abelian 11-extension of \mathbf{Q} .

For a specific example, consider $F = \mathbf{Q}(\zeta_{23})^+$; it is a cyclic 11-extension of \mathbf{Q} . We can easily use Theorem 4.1 to compute its λ -invariant: using that $\tau(23) = 18643272$ one finds that $h_{23}(\Delta) = 0$, so that $\lambda(\mathbf{Q}(\zeta_{23})^+, \Delta) = 11$.

For a more interesting example, take F to be the unique subfield of $\mathbf{Q}(\zeta_{1123})$ which is cyclic of order 11 over \mathbf{Q} . In this case we have

$$\tau(1123) \equiv 2 \pmod{11}$$

so that we have $h_{1123}(\Delta) = 20$. Thus, in this case, Theorem 4.1 shows that $\lambda(F, \Delta) = 31$.

4.3. The supersingular case. As mentioned in the introduction, the underlying principle of this paper is that the existence of a formula relating the λ -invariants of congruent Galois representations should imply a Kida-type formula for these invariants. We illustrate this now in the case of modular forms of weight two that are supersingular at p .

Let f be an eigenform of weight 2 and level N with Fourier coefficients in K some finite extension of \mathbf{Q}_p . Assume further than $p \nmid N$ and that $a_p(f)$ is not a p -adic

unit. In [8], Perrin-Riou associates to f a pair of algebraic and analytic μ -invariants over \mathbf{Q}_∞ which we denote by $\mu_\pm^*(\mathbf{Q}_\infty, f)$. (Here \star denotes either “alg” or “an” for algebraic and analytic respectively.) Moreover, when $\mu_+^*(\mathbf{Q}_\infty, f) = \mu_-^*(\mathbf{Q}_\infty, f)$ or when $a_p(f) = 0$, she also defines corresponding λ -invariants $\lambda_\pm^*(\mathbf{Q}_\infty, f)$. When $a_p(f) = 0$ these invariants coincide with the Iwasawa invariants of [6] and [9]. We also note that in [8] only the case of elliptic curves is treated, but the methods used there generalize to weight two modular forms.

We extend the definition of these invariants to the cyclotomic \mathbf{Z}_p -extension of an unramified abelian extension F of \mathbf{Q} . We define

$$\mu_\pm^*(F_\infty, f) = \sum_{\psi \in \text{Gal}(F/\mathbf{Q})^\vee} \mu_\pm^*(\mathbf{Q}_\infty, f_\psi) \quad \text{and} \quad \lambda_\pm^*(F_\infty, f) = \sum_{\psi \in \text{Gal}(F/\mathbf{Q})^\vee} \lambda_\pm^*(\mathbf{Q}_\infty, f_\psi)$$

for $\star \in \{\text{alg}, \text{an}\}$.

The following transition formula follows from the congruence results of [2].

Theorem 4.3. *Let f be as above and consider a p -extension of number fields F'/F with F'/\mathbf{Q} unramified at p . If $\mu_\pm^*(F_\infty, f) = 0$, then $\mu_\pm^*(F'_\infty, f) = 0$. Moreover, if this is the case, then*

$$\lambda_\pm^*(F'_\infty, f) = [F'_\infty : F_\infty] \cdot \lambda_\pm^*(F_\infty, f) + \sum_{w' \in R(F'_\infty/F_\infty)} m(F'_{\infty, w'} / F_{\infty, w}, V_f).$$

In particular, if the main conjecture is true for f over F (with $\mu_\pm^(F_\infty, f) = 0$), then the main conjecture is true for f over F' (with $\mu_\pm^*(F'_\infty, f) = 0$).*

Proof. The proof of this theorem proceeds along the lines of the proof of Theorem 3.1 replacing the appeals to the results of [1, 11] to the results of [2]. The main result of [2] is a formula relating the λ_\pm^* -invariants of congruent supersingular weight two modular forms. This formula has the same shape as the formulas that appear in [1] and [11] which allows for the proof to proceed nearly verbatim. \square

REFERENCES

- [1] M. Emerton, R. Pollack and T. Weston, *Variation of Iwasawa invariants in Hida families*, Invent. Math. **163**, (2006) no. 3, 523–580.
- [2] R. Greenberg, A. Iovita and R. Pollack, *Iwasawa invariants of supersingular modular forms*, preprint.
- [3] Y. Hachimori and K. Matsuno, *An analogue of Kida’s formula for the Selmer groups of elliptic curves*, J. Algebraic Geom. **8** (1999), no. 3, 581–601.
- [4] K. Iwasawa, *Riemann–Hurwitz formula and p -adic Galois representations for number fields*, Tohoku Math. J. **33** (1981), 263–288.
- [5] Y. Kida, *ℓ -extensions of CM-fields and cyclotomic invariants*, J. Number Theory **12** (1980), 519–528.
- [6] S. Kobayashi, *Iwasawa theory for elliptic curves at supersingular primes*, Invent. Math. **152** (2003), 1–36.
- [7] K. Matsuno, *An analogue of Kida’s formula for the p -adic L -functions of modular elliptic curves*, J. Number Theory **84** (2000), 80–92.
- [8] B. Perrin-Riou, *Arithmétique des courbes elliptiques à réduction supersingulière en p* , Experiment. Math. **12** (2003), no. 2, 155–186.
- [9] R. Pollack, *On the p -adic L -function of a modular form at a supersingular prime*, Duke Math. J. **118** (2003), no. 3, 523–558.
- [10] W. Sinnott, *On p -adic L -functions and the Riemann–Hurwitz genus formula*, Comp. Math. **53** (1984), 3–17.
- [11] T. Weston, *Iwasawa invariants of Galois deformations*, Manuscripta Math. **118** (2005) no. 2, 161–180.

- [12] K. Wingberg, *A Riemann–Hurwitz formula for the Selmer group of an elliptic curve with complex multiplication*, *Comment. Math. Helv.* **63** (1988), 587–592.

(Robert Pollack) DEPARTMENT OF MATHEMATICS, BOSTON UNIVERSITY, BOSTON, MA

(Tom Weston) DEPT. OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MA
E-mail address, Robert Pollack: `rpollack@math.bu.edu`
E-mail address, Tom Weston: `weston@math.umass.edu`