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Mazur-Tate elements of nonordinary modular forms. (English summary)
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The article under review is concerned with Iwasawa invariants of Mazur-Tate elements of cuspidal non-ordinary Hecke eigenforms. More precisely, the set-up is as follows. Let p be a prime and f be a cuspidal Hecke eigenform on $\Gamma_0(N)$ with $p\nmid N$ for some weight k such that the attached residual modulo p Galois representation $\overline{\rho}_f\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible of Serre weight 2 and its restriction to a decomposition group at p is indecomposable. The article deals with such f that are p-non-ordinary (i.e. the T_p-eigenvalue $a_p(f)$ on f is not a p-adic unit) in two cases:

(1) 'medium weight', i.e. $2<k<p^2+1$;

(2) 'low slope', i.e. $0<\mathrm{ord}_p(a_p(f))<p-1$.

In both cases there is a weight 2 eigenform g on $\Gamma_0(N)$ which is congruent to f modulo p (at coefficients away from p), implying $\overline{\rho}_f\cong\overline{\rho}_g$.

To the modular form f, one attaches so-called Mazur-Tate elements $\Theta_n(f)\in\mathbb{Z}_p[G_n]$, where G_n is the Galois group of the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Those are defined through a modular symbol associated with f. It is in terms of Mazur-Tate elements that the Iwasawa λ- and μ-invariants are defined.

The main theorems of the article relate the λ- and μ-invariants of the Mazur-Tate elements $\Theta_n(f)$ to the λ- and μ-invariants of g. More precisely, concerning λ, both in the medium weight and the low slope case, the authors express $\lambda(\Theta_n(f))$ by a simple formula in n and $\lambda(g)$ (if g is p-ordinary) or $\lambda^{\pm}(g)$ (if g is p-non-ordinary). In the medium weight case their result on the μ-invariant is that $\mu(\Theta_n(f))$ vanishes for large n if and only if $\mu(g)$ (if g is p-ordinary) or $\mu^{\pm}(g)$ (if g is p-non-ordinary) vanishes. In the low slope case, $\mu(\Theta_n(f))$ is equal to $\mu_{\min}(f)$ for large n if and only if $\mu(g)$ (if g is p-ordinary) or $\mu^{\pm}(g)$ (if g is p-non-ordinary) vanishes. Here $\mu_{\min}(f)$ is a certain naturally defined lower bound.

The authors also include an example for which the conclusions of their main results do not hold. For a sketch of the proofs the reader is referred to the introduction of the paper, where the authors give a very clear overview and sketch of the ingredients and the proofs.

Reviewed by *Gabor Wiese*

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