MR1983040 (2004e:11050) 11F67 (11R23)
Pollack, Robert [Pollack, Robert[2]] (1-CHI)
On the p-adic L-function of a modular form at a supersingular prime. (English summary)
Duke Math. J. **118** (2003), *no. 3,* 523–558.

Let E denote a modular elliptic curve over $\mathbf{Q}$, corresponding to a holomorphic cusp form f of weight 2, and let p denote a prime number. When E has good, ordinary reduction at p, then work of B. Mazur and P. Swinnerton-Dyer [Invent. Math. **25** (1974), 1–61; MR0354674 (50 #7152)] showed how to construct a p-adic L-function $L_p(E,T)=L_p(f,T)$ for f and E. Here T denotes a variable. It turns out that, in this ordinary case, $L_p(E,T)$ may be identified with a power series in $\mathbf{Z}_p[[T]]\otimes\mathbf{Q}_p$. Furthermore, Mazur's work on the behavior of the Selmer groups and Mordell-Weil groups of E in the cyclotomic $\mathbf{Z}_p$-extension $\mathbf{Q}_\infty/\mathbf{Q}$ gave a conjectural relationship between $L_p(E,T)$ and the Selmer group $S_\infty(E)$ of E over the field $\mathbf{Q}_\infty$ [Invent. Math. **18** (1972), 183–266; MR0444670 (56 #3020)]. Roughly speaking, Mazur conjectured that the power series $L_p(E,T)$ generates the characteristic of the Selmer group $S_\infty(E)$. This conjecture has recently been proven in unpublished work of Kato, Skinner, and Urban.

The construction of Mazur and Swinnerton-Dyer was generalized to higher weight forms and supersingular primes by a number of authors. For ordinary primes p, the L-function is once again a power series in $R[[T]]\otimes\mathbf{Q}_p$, where R is the ring of integers in a finite extension of $\mathbf{Q}_p$. In particular, these L-functions all have only finitely many zeroes, by the Weierstrass preparation theorem.

However, the supersingular case is quite different: the L-function has unbounded coefficients, and infinitely many zeroes. In fact, there are two different L-functions $L_p(E,\alpha,T)$ and $L_p(E,\beta,T)$ for each supersingular p, corresponding to the roots α and β of the Hecke polynomial $X^2-a_pX+p^{k-1}$. It is known (under various hypotheses) that at least one of these two L-functions must have infinitely many zeroes. The arithmetic nature of these zeroes remains quite mysterious. In particular, there is no plausible link between these p-adic L-functions and any Selmer group.

In the present paper, the author investigates the p-adic L-functions in the extreme case of a modular form f_k for which the number a_p is zero. This includes the case of an elliptic curve with good supersingular reduction, at least if $p>3$. In this review we restrict to the case of such elliptic curves.

The key observation is that the assumption $a_p=0$ implies that roots of the Hecke polynomial satisfy $\alpha=-\beta$. Using this fact, together with the interpolation property of $L_p(E,\alpha,T)$ and $L_p(E,\beta,T)$, the author defines power series $G^\pm(T)$ by the equation

$$L_p(E,\alpha,T)=G^+(T)+\alpha G^-(T),$$

and shows that the power series $G^\pm(T)$ are forced to have zeroes at certain prescribed roots of unity of p-power order. When p is odd, it turns out that $G^+(T)$ vanishes at all ζ_n with n even, while $G^-(X)$ vanishes at the remaining ζ_n with n odd. Here ζ_n denotes any primitive p^n-th root of unity.

On the other hand, it is known that the p-adic logarithm vanishes at all ζ_n. Thus the author is led to construct certain functions $\log^{\pm}(X)$ which vanish precisely at the trivial zeroes of $G^{\pm}(X)$. He then considers the ratios

$$L^{\pm}(E,X) = G^{\pm}(X)/\log^{\pm}(X)$$

and shows that each function $L^{\pm}(E,X)$ has only finitely many zeroes and has integral coefficients.

That these are the correct L-functions to consider has been demonstrated in recent work of S. Kobayashi [Invent. Math. **152** (2003), no. 1, 1–36; MR1965358 (2004b:11153)]. Indeed, Kobayashi defines two restricted Selmer groups $S^{\pm}_{\infty}(E)$ inside $S_{\infty}(E)$, and proves, using Kato's Euler system, that these Selmer groups $S^{\pm}_{\infty}(E)$ are annihilated by $L^{\pm}(E,T)$ respectively.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.