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Two p-adic L-functions and rational points on elliptic curves with supersingular reduction.

L-functions and Galois representations, 300–332, *London Math. Soc. Lecture Note Ser.*, 320, *Cambridge Univ. Press, Cambridge*, 2007.

In Iwasawa theory for elliptic curves, the case of supersingular reduction at p is usually regarded to be more complicated than the ordinary case. However, as the second author showed [Duke Math. J. **118** (2003), no. 3, 523–558; MR1983040 (2004e:11050)], the p-adic L-function $\mathfrak{L}_{p,\alpha}(E)$ of the curve E can be written as

$$\mathfrak{L}_{p,\alpha}(E) = f\log_p^+ + g\log_p^-\alpha$$

by using two Iwasawa functions f and g in $\mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]]$. Here E is an elliptic curve over $\mathbf{Q}$ with good supersingular reduction at the prime p, $\log^\pm$ is the $\pm$-log function, and $\mathbf{Q}_\infty/\mathbf{Q}$ is the cyclotomic $\mathbf{Z}_p$-extension. The goal of this paper is to present such examples where these nice and numerically computable Iwasawa functions f and g give us some advantage.

The first application is towards the weak Birch and Swinnerton-Dyer (BSD) conjecture. The authors prove that under some technical condition—which should always be satisfied—and the finiteness of the p-primary part $\text{Ш}(E/\mathbf{Q})[p^\infty]$ of the Tate-Shafarevich group we have

$$\operatorname{rank} E(\mathbf{Q}) > 0 \Longleftrightarrow \frac{f(T)}{g(T)}\Big|_{T=0} \neq \frac{p-1}{2}.$$

Here the left-hand side is algebraic information and the right-hand side is p-adic analytic information that can be numerically computed. This makes the weak BSD conjecture easier to verify numerically in various examples. The implication $\Leftarrow$ was essentially known before [cf. B. Perrin-Riou, Experiment. Math. **12** (2003), no. 2, 155–186; MR2016704 (2005h:11138)], but is also re-proven by the authors in a somewhat simpler way.

Next, the authors use the computation of $f'(0) - \frac{p-1}{2}g'(0)$ to produce rational points on the curve numerically whenever $\mathrm{ord}_{T=0}f(T) = \mathrm{ord}_{T=0}g(T) = 1$. Moreover, they interpret this value by using the p-adic BSD conjecture.

In the last section the authors study the so-called “fine Selmer group” and a problem raised by Greenberg concerning its characteristic element. They formulate a question regarding the greatest common divisor of the two p-adic L-functions f and g that would give an affirmative answer to Greenberg’s problem whenever the μ-invariant vanishes. Moreover, they also give a sufficient condition for the implication in the other direction.

{For the entire collection see MR2367390 (2008m:11003)}

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