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Variation of Iwasawa invariants in Hida families.
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The aim of this important paper is to study the variation of Iwasawa invariants of modular forms in Hida families.

Let f be a modular form and let p be a prime at which f is ordinary and p-distinguished. One can associate to f two kinds of p-adic power series. The first, $L_f^{\rm an}$, is the p-adic L-function, which is designed to interpolate the special values of the usual L-function associated to f. The second, $L_f^{\rm alg}$, is the characteristic series of the dual of the Selmer group of f. It is a generalization of a conjecture of B. Mazur [cf. Invent. Math. **18** (1972), 183–266; MR0444670 (56 #3020)] that $L_f^{\rm alg}$ and $L_f^{\rm an}$ are equal up to a unit: this is the "main conjecture of Iwasawa theory". K. Katō has proved in many cases [cf. Astérisque No. 295 (2004), ix, 117–290; MR2104361 (2006b:11051)] that $L_f^{\rm alg}$ divides $L_f^{\rm an}$ up to a power of p. For $* \in \{{\rm alg}, {\rm an}\}$, let $\mu^*(f)$ be the exponent of the highest power of p dividing L_f^* and let $\lambda^*(f)$ be the number of zeroes of L_f^*. Katō's result implies that

$$\mu^{\rm alg}(f) \le \mu^{\rm an}(f) \text{ and } \lambda^{\rm alg}(f) \le \lambda^{\rm an}(f)$$

and the main conjecture is equivalent to equality in both cases. It therefore represents important progress towards the main conjecture to be able to prove something about those invariants, and this is what the authors do in this paper.

Let $\overline{\rho}$ be the mod p representation attached to f and let $\mathcal{H}(\overline{\rho})$ be the Hida family of $\overline{\rho}$, that is, the set of all p-ordinary p-stabilized newforms f with mod p Galois representation isomorphic to $\overline{\rho}$. These newforms are dense in a p-adic analytic space of overconvergent eigenforms, consisting of an intersecting system of branches $\mathbf{T}(\mathfrak{a})$ indexed by the minimal primes $\mathfrak{a}$ of a certain Hecke algebra. The main results of the paper concern the variation of the invariants $\mu^*(f)$ and $\lambda^*(f)$ as f varies in $\mathcal{H}(\overline{\rho})$. For $* \in \{{\rm alg}, {\rm an}\}$, the two main theorems are:

Theorem 1. If $\mu^*(f_0) = 0$ for some $f_0 \in \mathcal{H}(\overline{\rho})$, then $\mu^*(f) = 0$ for all $f \in \mathcal{H}(\overline{\rho})$ (we then write $\mu^*(\overline{\rho}) = 0$).

Theorem 2. If $\mu^*(\overline{\rho}) = 0$, and if f_1 and f_2 lie on two branches $\mathbf{T}(\mathfrak{a}_1)$ and $\mathbf{T}(\mathfrak{a}_2)$, then $\lambda^*(f_1) - \lambda^*(f_2) = \sum_{l|N_1N_2} e_l(\mathfrak{a}_2) - e_l(\mathfrak{a}_1)$ where $e_l(\mathfrak{a})$ is some explicit invariant of $\mathbf{T}(\mathfrak{a})$.

We refer to the article's informative introduction for an explanation of the proofs and for some corollaries of Theorems 1 and 2 above. The last section (§5.3) of the article consists of numerical examples and some open questions.

Reviewed by *Laurent N. Berger*

References

1. Carayol, H.: Sur les représentations l-adiques associées aux formes modulaires de Hilbert. Ann. Sci. Éc. Norm. Supér., IV. Sér. **19,** 409–468 (1986) MR0870690 (89c:11083)
2. Carayol, H.: Formes modulaires et représentations galoisiennes à valeurs dans un anneau local complet. p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), Contemp. Math., vol. 165, pp. 213–237. Providence, RI: Am. Math. Soc. 1994 MR1279611 (95i:11059)
3. Darmon, H., Diamond, F., Taylor, R.: Fermat's last theorem. Elliptic curves, modular forms & Fermat's last theorem (Hong Kong, 1993), pp. 2–140. Cambridge, MA: Internat. Press 1997
4. Diamond, F.: On congruence modules associated to Λ-adic forms. Compos. Math. **71,** 49–83 (1989) MR1008804 (90h:11041)
5. Diamond, F.: The refined conjecture of Serre. Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993). Ser. Number Theory, I, pp. 22–37. Cambridge, MA: Internat. Press 1995 MR1363493 (97b:11065)
6. Diamond, F.: On deformation rings and Hecke rings. Ann. Math. (2) **144,** 137–166 (1996) MR1405946 (97d:11172)
7. Diamond, F., Ribet, K.A.: l-adic modular deformations and Wiles's "main conjecture". Modular forms and Fermat's last theorem (Boston, MA, 1995), pp. 357–371. New York: Springer 1997 MR1638485
8. Diamond, F., Taylor, R.: Nonoptimal levels of $\mod l$ modular representations. Invent. Math. **115,** 435–462 (1994) MR1262939 (95c:11060)
9. Diamond, F., Taylor, R.: Lifting modular $\mod l$ representations. Duke Math. J. **74,** 253–269 (1994) MR1272977 (95e:11052)
10. Greenberg, R.: Iwasawa theory for p-adic representations. Algebraic number theory, Adv. Stud. Pure Math., vol. 17, pp. 97–137. Boston, MA: Academic Press 1989 MR1097613 (92c:11116)
11. Greenberg, R.: Iwasawa theory for motives. In L-functions and arithmetic (Durham, 1989), pp. 211–233. Cambridge: Cambridge Univ. Press 1991 MR1110394 (93a:11092)
12. Greenberg, R.: Iwasawa theory and p-adic deformations of motives. Motives (Seattle, WA, 1991). Proc. Symp. Pure Math., vol. 55, pp. 193–223. Providence, RI: Am. Math. Soc. 1994 MR1265554 (95i:11053)
13. Greenberg, R.: Iwasawa theory for elliptic curves. Arithmetic theory of elliptic curves (Cetraro, 1997). Lect. Notes Math., vol. 1716, pp. 51–144. Berlin: Springer 1999 MR1754686 (2002a:11056)
14. Greenberg, R., Stevens, G.: p-adic L-functions and p-adic periods of modular forms. Invent. Math. **111,** 407–447 (1993) MR1198816 (93m:11054)
15. Greenberg, R., Vatsal, V.: On the Iwasawa invariants of elliptic curves. Invent. Math. **142,** 17–63 (2000) MR1784796 (2001g:11169)
16. Gross, B.: A tameness criterion for Galois representations associated to modular forms $(\mod p)$. Duke Math. J. **61,** 445–517 (1990) MR1074305 (91i:11060)
17. Hida, H.: Galois representations into $\mathrm{GL}_2(\mathbf{Z}_\mathrm{p}[[\mathrm{X}]])$ attached to ordinary cusp forms. Invent. Math. **85,** 545–613 (1986) MR0848685 (87k:11049)
18. Hida, H.: Iwasawa modules attached to congruences of cusp forms. Ann. Sci. Éc. Norm. Supér., IV. Sér. **19,** 231–273 (1986) MR0868300 (88i:11023)

19. Hida, H.: Modules of congruence of Hecke algebras and L-functions associated with cusp forms. Am. J. Math. **110,** 323–382 (1988) MR0935010 (89i:11058)
20. Kato, K.: p-adic Hodge theory and values of zeta functions of modular forms. Cohomologies p-adiques et applications arithmétiques. III. Astérisque **295,** 117–290 (2004) MR2104361 (2006b:11051)
21. Katz, N.: Higher congruences between modular forms. Ann. Math. (2) **101,** 332–367 (1975) MR0417059 (54 #5120)
22. Kitigawa, K.: On standard p-adic L-functions of elliptic curves. In: p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991). Contemp. Math., vol. 165, pp. 81–110. Providence, RI: Am. Math. Soc. 1994
23. Livné, R.: On the conductors of $\mod l$ Galois representations coming from modular forms. J. Number Theory **31,** 133–141 (1989) MR0987567 (90f:11029)
24. Matsumura, H.: Commutative ring theory, second edn. Cambridge Studies in Advanced Mathematics, vol. 8. Translated from the Japanese by M. Reid. Cambridge: Cambridge University Press 1989 MR1011461 (90i:13001)
25. Mazur, B.: On the arithmetic of special values of L-functions. Invent. Math. **55,** 207–240 (1979) MR0553997 (82e:14033)
26. Mazur, B.: Euler systems in arithmetic geometry. Notes from a course at Harvard University (1998), available at http://www.math.amherst.edu/∼taweston
27. Mazur, B.: The theme of p-adic variation. Mathematics: frontiers and perspectives, pp. 433–459. Providence, RI: Am. Math. Soc. 2000 MR1754790 (2001i:11064)
28. Mazur, B.: Two-variable p-adic L-functions. Unpublished notes
29. Mazur, B.: An introduction to the deformation theory of Galois representations. In: Modular forms and Fermat's last theorem (Boston, MA, 1995), pp. 243–311. New York: Springer 1997 MR1638481
30. Ochiai, T.: On the two-variable Iwasawa main conjecture for Hida deformations. In preparation
31. Panchishkin, A.: Two-variable p-adic L-functions attached to eigenfamilies of positive slope. Invent. Math. **154,** 551–615 (2003) MR2018785 (2004k:11065)
32. Stein, W.: The modular forms explorer. http://modular.fas.harvard.edu/mfd/mfe/
33. Stevens, G.: Families of overconvergent modular symbols. Preprint
34. Taylor, R., Wiles, A.: Ring-theoretic properties of certain Hecke algebras. Ann. Math. (2) **141,** 553–572 (1995) MR1333036 (96d:11072)
35. Vatsal, V.: Canonical periods and congruence formulae Duke Math. J. **98,** 397–419 (1999) MR1695203 (2000g:11032)
36. Weston, T.: Iwasawa invariants of Galois deformations. Manuscr. Math. **118,** 161–180 (2005) MR2177683 (2006k:11211)
37. Wiles, A.: Modular elliptic curves and Fermat's last theorem. Ann. Math. (2) **141,** 443–551 (1995) MR1333035 (96d:11071)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.