ON THE FREENESS OF ANTICYCLOTOMIC SELMER GROUPS

CHAN-HO KIM, ROBERT POLLACK, AND TOM WESTON

Abstract. We establish the freeness of certain anticyclotomic Selmer groups of modular forms. The freeness of these Selmer groups plays a key role in the Euler system arguments introduced in [BD05]. In particular, our result fills some implicit gaps in [PW11] and [CH15] which in turn allows the results of these papers to hold for modular forms whose residual representations are not minimally ramified. Removing these minimal ramification conditions is essential for applications of congruences of modular forms to anticyclotomic Iwasawa theory as in [PW11, §7] and [Kim].

1. Introduction

In [BD05], Bertolini and Darmon study the Iwasawa theory of an elliptic curve $E/\mathbb{Q}$ up the anticyclotomic $\mathbb{Z}_p$-extension of a quadratic imaginary field $K/\mathbb{Q}$. They consider the case when there are an *odd* number of bad primes for E which are inert in K. This assumption prevents the existence of Heegner points for E up this tower and thus allows for the Selmer group of E to be cotorsion over the Iwasawa algebra. In this case, one divisibility of the anticyclotomic main conjecture is proven: namely, the characteristic power series of the Selmer group divides the p-adic L-function. Despite the lack of Heegner points for E, the method of proof in [BD05] is via an Euler system constructed out of Heegner points arising from a family of forms highly congruent to the modular form associated to E (but with additional inert primes in their levels).

In [PW11], this anticyclotomic divisibility was generalized both to weight 2 modular forms with arbitrary Fourier coefficients and to forms satisfying some less stringent ramification assumptions on their residual representations. However, in this generalization, an implicit error was introduced. The freeness of certain non-primitive anticyclotomic Selmer groups as Galois modules (see [BD05, Proposition 3.3]), which played a key role in the Euler system argument of [BD05], no longer clearly holds in this setting. We also note that in [CH15] the arguments of [PW11] were further generalized to higher weight modular forms, and this same error leaked into their work as well.

The main goal of this article is to give an alternate proof of the freeness of these non-primitive anticyclotomic Selmer group which works in a more general setting than [BD05, Proposition 3.3]. In particular, this argument fixes the errors in both [PW11] and [CH15]. Further, the argument given here is a bit novel (especially section 3), and may have use in Iwasawa theory in other contexts.

1.1. **Statement of the main theorem.** Fix a prime $p > 3$, k a weight satisfying $2 \leq k < p-1$, and $f = \sum a_n q^n \in \overline{\mathbb{Q}}_p[\![q]\!]$ a p-ordinary newform of weight k on $\Gamma_0(N)$ with $p \nmid N$. Let $E = \mathbb{Q}_p(\{a_n\})$, $\mathcal{O} := \mathcal{O}_E$, and $\mathbb{F} = \mathcal{O}/\varpi\mathcal{O}$ where ϖ is a uniformizer of $\mathcal{O}$.

Let K be an imaginary quadratic field with $(\mathrm{disc}(K), Np) = 1$ so that we can decompose $N = N^+ \cdot N^-$ where $\ell | N^+$ (resp. $\ell | N^-$) if and only if ℓ splits in K (resp. ℓ is inert in K). We assume that N^- is the squarefree product of an odd number of primes.

Let $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(E)$ denote the Galois representation associated to f with residual representation $\overline{\rho}_f$. Here we are taking the representation coming from homology so that $\det(\rho_f) = \varepsilon^{k-1}$ where ε is the cyclotomic character. We also consider the self-dual twist of ρ_f:

$$\rho_f^* := \rho_f \otimes \varepsilon^{\frac{2-k}{2}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(V_f).$$

Note that V_f is the underlying vector space of ρ_f^* and will be the main representation we examine throughout this paper.

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Let T_f denote some Galois-stable $\mathcal{O}$-lattice in V_f, and set $A_f = V_f/T_f$. Define $T_{f,n} := T_f/\varpi^n T_f$ and $A_{f,n} := A_f[\varpi^n]$. Note that $T_{f,n} \simeq A_{f,n}$ and so the notational difference is just suggestive of the standard compatibilities with inverse and direct limits respectively. Further, set $\overline{\rho}_f^* \cong T_{f,1}$ so that $\overline{\rho}_f^* \cong \overline{\rho} \otimes \overline{\varepsilon}^{\frac{2-k}{2}}$ with $\overline{\varepsilon}$ the mod p reduction of the cyclotomic character.

We make a careful listing here of the various hypotheses that will be needed throughout the paper. The first two hypotheses below match those in [PW11] and [CH15], and thus depend on whether or not $k = 2$. The third is needed to apply the Euler system argument.

Assumption 1.1. $(k = 2)$

(1) *$\overline{\rho}_f$ is irreducible.*
(2) *$\overline{\rho}_f$ is ramified at ℓ if $\ell \mid N^-$ and $\ell \equiv \pm 1 \pmod p$*
(3) *$a_p(f) \not\equiv \pm 1 \pmod p$.*

Assumption 1.2. $(2 < k < p-1)$

(1) *The restriction of $\overline{\rho}_f$ to the absolute Galois group of $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$ is absolutely irreducible.*
(2) *$\overline{\rho}_f$ is ramified at ℓ if either*
   - *$\ell \mid N^-$ and $\ell \equiv \pm 1 \pmod p$ or*
   - *$\ell \| N^+$ and $\ell \equiv 1 \pmod p$.*
(3) *The restriction of $\overline{\rho}_f$ to the inertia group of $\mathbb{Q}_\ell$ is irreducible if $\ell^2 \mid N$ and $\ell \equiv -1 \pmod p$.*

Assumption 1.3. *One of the following conditions holds:*

(1) $\#\left((\mathbb{F}_p^\times)^{k-1}\right) > 5$
(2) *The image of $\overline{\rho}_f^*$ contains a conjugate of $\mathrm{GL}_2(\mathbb{F}_p)$.*

Remark 1.4. The hypotheses in Assumption 1.1 nearly corresponds to "Hypothesis CR" in [PW11] except that the assumption $a_p(f) \not\equiv \pm 1 \pmod p$ was erroneously not included in [PW11]. This assumption is needed to ensure that f is not congruent to a a newform of level Np which in turn allows for the needed application of Ihara's lemma in the Euler system argument (see [CH15, Lemma 5.3]). The hypotheses in Assumption 1.2 are identical to the assumptions required in [CH15].

To simplify the statements of our main results, we combine these assumptions into a single one which we will refer to as Assumption (A):

- If $k = 2$, Assumption 1.1 holds.
- If $k > 2$, Assumption 1.2 holds.
- Assumption 1.3 holds.

Let K_∞ be the anticyclotomic $\mathbb{Z}_p$-extension of K with Galois groups Γ_∞. Set K_m equal to the m-th layer of this extension with Galois group G_m. Let $\Lambda = \mathcal{O}[\![\Gamma_\infty]\!]$ be the anticyclotomic Iwasawa algebra. Let S be a set of n-admissible primes as in [CH15, Definition 1.1] and let $\mathrm{Sel}^{SN^+}(K_m, A_{f,n})$ denote the non-primitive Selmer group which consists of classes in $H^1(K_m, A_{f,n})$ which are:

- unramified outside NSp,
- ordinary at $\ell \mid N^-p$, and
- arbitrary at $\ell \mid SN^+$

as in [CH15, Definition 1.2]. Taking the inverse limit under corestriction, we define

$$\widehat{\mathrm{Sel}}^{SN^+}(K_\infty, T_{f,n}) := \varprojlim_m \mathrm{Sel}^{SN^+}(K_m, T_{f,n}).$$

We also define

$$\mathrm{Sel}^{SN^+}(K_\infty, A_f) := \varinjlim_{m,n} \mathrm{Sel}^{SN^+}(K_m, A_{f,n}).$$

The following is our main theorem.

Theorem 1.5. *Under Assumption (A), $\widehat{\mathrm{Sel}}^{SN^+}(K_\infty, T_{f,n})$ is a free $\Lambda/\varpi^n\Lambda$-module of rank $\#S$.*

The proof will be given in §4.

Remark 1.6. (1) We again remark that this theorem fills the gap in [PW11] (and thus [CH15]) as discussed at the start of the introduction.

(2) Theorem 1.5 would also hold for non-ordinary forms of weight 2 satisfying the conditions in [PW11, Theorem 5.3.2] if we replace the ordinary Selmer group with the corresponding plus/minus Selmer group as in [IP06]. Since the arguments in the non-ordinary case are essentially identical, we do not include them here.

1.2. **Strategy of the proof.** The following hypothesis was implicitly assumed in [PW11], and we now explain how we aim to side step this assumption.

Assumption 1.7. *(N^+-minimality) $\overline{\rho}_f$ is ramified at all primes dividing N^+.*

Under Assumption 1.7, the finite level Selmer group $\widehat{\mathrm{Sel}}^{SN^+}(K_m, T_{f,n})$ is a free $(\mathcal{O}/\varpi^n\mathcal{O})[G_m]$-module (see [BD94, Theorem 3.2], [CH15, Corollary 6.9]) where $G_m = \mathrm{Gal}(K_m/K)$. In particular, Theorem 1.5 follows immediately in this case by taking inverse limits. In particular, the results of [PW11] and [CH15] hold as stated in this case.

Unfortunately, without Assumption 1.7, these finite level Selmer groups need not be free. Nonetheless, we show that the inverse limit of these modules will be free even if freeness fails at finite levels. Indeed, standard control theorems (see Lemma 2.1) tell us that

$$\mathrm{Sel}^{SN^+}(K_m, A_{f,n}) \simeq \mathrm{Sel}^{SN^+}(K_\infty, A_f)[\varpi^n]^{\Gamma_m}.$$

Note that the groups $\left\{\mathrm{Sel}^{SN^+}(K_\infty, A_f)[\varpi^n]^{\Gamma_m}\right\}$ naturally form a directed system in m under inclusion. However, these groups also form an *inverse* system in m under the trace maps. Moreover, by definition, their inverse limit is isomorphic to $\widehat{\mathrm{Sel}}^{SN^+}(K_\infty, T_{f,n})$. In section 3, we show for an abstract cofinitely generated Λ-module A that $\varprojlim_m A[\varpi^n]^{\Gamma_m}$ depends only on the $\mathcal{O}$-torsion in $A^\vee = \mathrm{Hom}_{\mathcal{O}}(A, E/\mathcal{O})$. Further, we give precise conditions on this $\mathcal{O}$-torsion to ensure that our inverse limit is free over $\Lambda/\varpi^n\Lambda$.

Thus, to prove Theorem 1.5 it suffices to control the $\mathcal{O}$-torsion of $\mathrm{Sel}^{SN^+}(K_\infty, A_f)^\vee$. The Galois cohomological arguments in [PW11, §3] apply equally well here to give a lower bound on this $\mathcal{O}$-torsion (see Proposition 2.4 and Lemma 2.5). The upper bound is more subtle and tantamount to the vanishing of the μ-invariant of $\mathrm{Sel}(K_\infty, A_f)^\vee$. This presents a problem as this vanishing of μ-invariants is proven in [PW11] and [CH15] via an Euler system argument, but these are exactly the argument we are attempting to fix in this paper!

Nonetheless, we have the following work-around: by level-lowering, we can find an eigenform g congruent to f and so that g satisfies Assumption 1.7. In this case, the results of [PW11] and [CH15] hold and we know that the μ-invariant of $\mathrm{Sel}(K_\infty, A_g)^\vee$ vanishes. Standard arguments as in [GV00] then give that the μ-invariant of $\mathrm{Sel}(K_\infty, A_f)^\vee$ vanishes as desired. See Corollary 2.3 for this argument, and Corollary 2.6 for the full description of the $\mathcal{O}$-torsion in $\mathrm{Sel}^{SN^+}(K_\infty, A_f)^\vee$.

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2. Selmer groups

2.1. **Control theorems.** The following two control theorems will be crucially used throughout.

Lemma 2.1. *We have*

$$\mathrm{Sel}^{SN^+}(K_m, A_{f,n}) \simeq \mathrm{Sel}^{SN^+}(K_\infty, A_{f,n})^{\Gamma_m}$$

and

$$\mathrm{Sel}^{SN^+}(K_\infty, A_{f,n}) \simeq \mathrm{Sel}^{SN^+}(K_\infty, A_f)[\varpi^n].$$

Proof. [CH15, Proposition 1.9] □

2.2. **Congruences: μ-invariants and cotorsionness.**

Theorem 2.2. *Under Assumption (A) and Assumption 1.7,* $\mathrm{Sel}(K_\infty, A_f)$ *is Λ-cotorsion and its μ-invariant vanishes.*

Proof. This is [PW11, Theorem 1.1 and 1.3] and [CH15, Corollary 1] depending on whether the weight k satisfies $k = 2$ or $k > 2$. As discussed in section 1.2, the results of these papers hold as stated under the additional Assumption 1.7. We do note that the conditions listed under Assumption (A), namely those in Assumption 1.3, are weaker than in [PW11] (where $\overline{\rho}_f$ is assumed to be surjective) and in [CH15] (where it is assumed that $\#\left((\mathbb{F}_p^\times)^{k-1}\right) > 5$). Nonetheless, the needed Euler system arguments of [CH15] go through under our slightly weaker assumptions (namely that if $\#\left((\mathbb{F}_p^\times)^{k-1}\right) \leq 5$, then the image of $\overline{\rho}_f$ contains a conjugate of $\mathrm{GL}_2(\mathbb{F}_p)$). To see this, one just needs to carefully follow through the details of [CH15, Lemmas 6.1 and 6.2] and see that these lemmas still hold in this case. □

In the following corollary, we remove Assumption 1.7 via level-lowering.

Corollary 2.3. *Under Assumption (A), we have that both* $\mathrm{Sel}(K_\infty, A_f)$ *and* $\mathrm{Sel}^{N^+}(K_\infty, A_f)$ *are* Λ*-cotorsion and their* μ*-invariants vanish.*

Proof. We reduce the statement to the case of Theorem 2.2. Applying level-lowering at primes dividing N^+, we can find a newform g which is congruent to f and satisfies all the assumptions of Theorem 2.2. Thus, $\mathrm{Sel}(K_\infty, A_g)$ is Λ-cotorsion and its μ-invariant vanishes. Next, the arguments in [GV00, Corollary 2.3] imply that $\mathrm{Sel}^{N^+}(K_\infty, A_g)$ is Λ-cotorsion and its μ-invariant vanishes. We note that in applying these arguments it is essential that all primes dividing N^+ split in K so that they are not infinitely split in K_∞.

Further, the final displayed equation in the proof of [PW11, Proposition 3.6] gives an exact sequence

$$0 \to \mathrm{Sel}(K_\infty, A_f[\varpi]) \to \mathrm{Sel}(K_\infty, A_f)[\varpi] \to \prod_{\substack{\ell \mid N^+ \\ w \in \sigma_{\infty,\ell}}} A_f^{G_{K_{\infty,w}}} / \varpi A_f^{G_{K_{\infty,w}}}.$$

Removing the local conditions at primes dividing N^+ then yields

$$\mathrm{Sel}^{N^+}(K_\infty, A_f)[\varpi] \simeq \mathrm{Sel}^{N^+}(K_\infty, A_g)[\varpi] \tag{1}$$

as both Selmer groups are isomorphic to $\mathrm{Sel}(K_\infty, \overline{\rho}^*)$ where $\overline{\rho}^* \simeq \overline{\rho}_f^* \simeq \overline{\rho}_g^*$.

Recall that a Λ-module S is cotorsion with vanishing μ-invariant if and only if $S[\varpi]$ is finite. Thus, the right hand side of (1) is finite, and hence so is the left. But then this finiteness implies that $\mathrm{Sel}^{N^+}(K_\infty, A_f)$ is Λ-cotorsion with vanishing μ-invariant. As $\mathrm{Sel}(K_\infty, A_f) \subseteq \mathrm{Sel}^{N+}(K_\infty, A_f)$, we see immediately that $\mathrm{Sel}(K_\infty, A_f)$ is Λ-cotorsion with vanishing μ-invariant. □

2.3. **Non-primitive Selmer groups.** Let

$$\mathcal{H}_\ell = \varinjlim_m \prod_{\substack{w \mid \ell \\ w \text{ in } K_m}} H^1(K_{m,w}, A_f).$$

Proposition 2.4. *Let* S *be a finite collection of primes which are disjoint from the divisors of* Np*. Under Assumption (A) we have the following exact sequence*

$$0 \to \mathrm{Sel}^{N^+}(K_\infty, A_f) \to \mathrm{Sel}^{SN^+}(K_\infty, A_f) \to \prod_{\ell \in S} \mathcal{H}_\ell \to 0.$$

Proof. The content here is to show the surjectivity of the final map, which, in turn, follows from the surjectivity of

$$H^1(K_\Sigma/K_\infty, A_f) \to \prod_{\ell \in \Sigma} \mathcal{H}_\ell$$

where Σ is any finite collection of primes containing the prime divisors of Np. This surjectivity is proven in [PW11, Proposition A.2] where the hypothesis that $\mathrm{Sel}(K_\infty, A_f)$ be Λ-cotorsion is given by Corollary 2.3. We note that we cannot directly invoke the similar [GV00, Proposition 2.1] here as that proposition does not allow for primes which split infinitely in K_∞/K. □

Lemma 2.5. *Let* ℓ *be an* n*-admissible prime (as in [CH15, Definition 1.1]). Then we have* $\mathcal{H}_\ell \simeq (\Lambda/\varpi^t\Lambda)^\vee$ *with* $t \geq n$*.*

Proof. Since ℓ is n-admissible, ℓ is inert in $K/\mathbb{Q}$ and splits completely in K_m/K. Thus, we have

$$H^1(K_{m,w}, A_f) \simeq H^1(K_\lambda, A_f) \otimes_{\mathcal{O}} \mathcal{O}[\mathrm{Gal}(K_m/K)] \tag{2}$$

where λ is the unique prime of K above ℓ. By inflation-restriction, we have

$$0 \to H^1(K_\lambda^{\mathrm{un}}/K_\lambda, A_f) \to H^1(K_\lambda, A_f) \to H^1(I_\lambda, A_f)^{\mathrm{Gal}(K_\lambda^{\mathrm{un}}/K_\lambda)} \to 0$$

since A_f is unramified at ℓ and $\mathrm{Gal}(K_\lambda^{\mathrm{un}}/K_\lambda) \simeq \widehat{\mathbb{Z}}$ has cohomological dimension 1.

To deal with the first term, we have

$$H^1(K_\lambda^{\mathrm{un}}/K_\lambda, A_f) \cong A_f(\overline{\mathbb{Q}}_\ell)/(\mathrm{Frob}_\lambda - 1)A_f(\overline{\mathbb{Q}}_\ell).$$

Note that

$$A_f(\overline{\mathbb{Q}}_\ell) \overset{\mathrm{Frob}_\lambda - 1}{\longrightarrow} A_f(\overline{\mathbb{Q}}_\ell)$$

has kernel equal to $A_f(K_\lambda)$ which is finite. Thus, this map is surjective as $A_f(\overline{\mathbb{Q}}_\ell)$ is divisible. In particular, $H^1(K_\lambda^{\mathrm{un}}/K_\lambda, A_f) = 0$.

To deal with the second term, we have

$$H^1(I_\lambda, A_f) \cong \mathrm{Hom}(I_\lambda, A_f) \cong \mathrm{Hom}(I_\lambda^{\mathrm{tame}}, A_f) \cong \mathrm{Hom}(\mathbb{Z}_p(1), A_f).$$

Thus, $H^1(I_\lambda, A_f)^{\mathrm{Gal}(K_\lambda^{\mathrm{un}}/K_\lambda)}$ is the Frob_λ-equivariant maps from $\mathbb{Z}_p(1)$ to A_f. To compute such maps, we list the relevant Frobenius actions on these modules:

- Frob_λ acts on $\mathbb{Z}_p(1)$ by scalar multiplication by ℓ^2
- Frob_λ acts on A_f by eigenvalues

$$(\alpha_\ell^*)^2 = \left(\ell^{\frac{2-k}{2}} \cdot \alpha_\ell\right)^2 = \ell^{2-k} \cdot \alpha_\ell^2$$

and

$$(\beta_\ell^*)^2 = \left(\ell^{\frac{2-k}{2}} \cdot \beta_\ell\right)^2 = \ell^{2-k} \cdot \beta_\ell^2$$

where α_ℓ and β_ℓ are the roots of the ℓ-th Hecke polynomial of f so that $\alpha_\ell^* + \beta_\ell^* = \ell^{\frac{2-k}{2}} \cdot a_\ell(f)$ and $\alpha_\ell^* \cdot \beta_\ell^* = \ell^{2-k} \cdot \ell^{k-1} = \ell$.

Since ℓ is n-admissible, we have $\ell^{\frac{k}{2}} + \ell^{\frac{k-2}{2}} \equiv \pm a_\ell(f) \pmod{\varpi^n}$, or equivalently that $\ell + 1 \equiv \pm \ell^{\frac{2-k}{2}} a_\ell(f) \pmod{\varpi^n}$. Thus, we may assume $\alpha_\ell^* \equiv \pm 1 \pmod{\varpi^n}$ and $\beta_\ell^* \equiv \pm \ell \pmod{\varpi^n}$ with the same sign. Since ℓ is n-admissible, $\ell^2 \not\equiv 1 \pmod{\varpi}$; hence, $(\alpha_\ell^*)^2$ and $(\beta_\ell^*)^2$ are distinct modulo ϖ.

We first show that $(\beta_\ell^*)^2 \neq \ell^2$ and thus $H^1(I_\lambda, A_f)$ is finite. To see this, assume $\beta_\ell^* = \pm\ell$ and thus $\alpha_\ell^* = \pm 1$ since $\alpha_\ell^* \beta_\ell^* = \ell$. Thus, $\alpha_\ell^* + \beta_\ell^* = \pm(1+\ell)$ and

$$a_\ell(f) = \pm \ell^{\frac{k-2}{2}}(\alpha_\ell^* + \beta_\ell^*) = \pm \ell^{\frac{k-2}{2}}(1+\ell) = \pm(\ell^{\frac{k-2}{2}} + \ell^{\frac{k}{2}}).$$

But this is a contradiction as the above equality violates the Deligne-Ramanujan-Petersson bound $\mid a_\ell(f) \mid \leq 2\ell^{\frac{k-1}{2}}$.

Hence, there is some maximal integer t such that $\beta_\ell^* \equiv \pm \ell \pmod{\varpi^t}$, and we have $H^1(K_\lambda, A_f) \simeq \mathcal{O}/\varpi^t\mathcal{O}$. Note that by the n-admissibility of ℓ, we have that $t \geq n$. By (2), $H^1(K_{m,w}, A_f) \simeq (\mathcal{O}/\varpi^t\mathcal{O})[\mathrm{Gal}(K_m/K)]$. Taking the direct limit over all m then gives the lemma. □

Corollary 2.6. *If S is a finite set of n-admissible primes, then $\mathrm{Sel}^{SN^+}(K_\infty, A_f)^\vee$ is pseudo-isomorphic to*

$$\left(\bigoplus_{i=1}^{\#S} \Lambda/\varpi^{t_i}\Lambda\right) \times Y$$

with each $t_i \geq n$ and Y a Λ-torsion module with $\mu(Y) = 0$. In particular,

$$\mu(\mathrm{Sel}^{SN^+}(K_\infty, A_f)^\vee) \geq n \cdot \#S.$$

Proof. Combining Corollary 2.3, Proposition 2.4, and Lemma 2.5, we have

$$0 \to \bigoplus_{i=1}^{\#S} \Lambda/\varpi^{t_i}\Lambda \to \mathrm{Sel}^{SN^+}(K_\infty, A_f)^\vee \to Y \to 0$$

with $t_i \geq n$ and Y a Λ-torsion module with $\mu(Y) = 0$. Write $X = \mathrm{Sel}^{SN^+}(K_\infty, A_f)^\vee$ for simplicity and $X_{\mathcal{O}\text{-tor}}$ for the $\mathcal{O}$-torsion module of X. Then

$$X_{\mathcal{O}\text{-tor}} \simeq \bigoplus_{i=1}^{\#S} \Lambda/\varpi^{t_i}\Lambda$$

and

$$X/X_{\mathcal{O}\text{-tor}} \simeq Y.$$

Since a finitely generated Λ-module X is pseudo-isomorphic to $X_{\mathcal{O}\text{-tor}} \times X/X_{\mathcal{O}\text{-tor}}$, we obtain the conclusion. □

3. A functor

Let S be a cofinitely generated Λ-module. We fix an integer $n \geq 1$, and consider the collection of modules $S[\varpi^n]^{\Gamma_m}$ as m varies. These modules naturally form a direct system under inclusion and

$$\varinjlim_m S[\varpi^n]^{\Gamma_m} = S[\varpi^n].$$

However, one can also naturally put these modules into an inverse system via the following trace maps

$$\pi_m^{m+1} : S[\varpi^n]^{\Gamma_{m+1}} \to S[\varpi^n]^{\Gamma_m}$$
$$s \mapsto \sum_{\gamma \in \Gamma_{m+1}/\Gamma_m} \gamma s.$$

Taking inverse limits allows us to define a functor

$$\Phi_n : \{\text{cofinitely generated } \Lambda\text{-modules}\} \to \{\text{finitely generated } \Lambda/\varpi^n\Lambda\text{-modules}\}$$

by

$$\Phi_n(S) := \varprojlim_m S[\varpi^n]^{\Gamma_m}.$$

3.1. **Basic properties of Φ_n.** We now establish the basic properties of this functor through a series of lemmas.

Lemma 3.1. *The functor Φ_n is covariant and left exact.*

Proof. The functor Φ_n is defined by taking torsion, invariants and inverse limits, each of which are covariant and left exact. □

Lemma 3.2. *Let S be a cofinitely generated Λ-module. Then $\Phi_n(S) = \Phi_n(S[\varpi^n])$.*

Proof. This lemma is immediate from the definition. □

Lemma 3.3. *Let S be a Λ-module with finite cardinality. Then $\Phi_n(S) = 0$ for all $n \geq 1$.*

Proof. Since S is finite, for m large enough, Γ_m fixes all elements of S. But then, for such m, the transition maps π_m^{m+1} defining $\Phi_n(S)$ are simply multiplication by p as

$$\pi_m^{m+1}(s) = \sum_{\gamma \in \Gamma_m/\Gamma_{m+1}} \gamma s = \sum_{\gamma \in \Gamma_m/\Gamma_{m+1}} s = ps.$$

Thus, the finiteness of S implies the vanishing of $\Phi_n(S)$. □

Lemma 3.4. *If $0 \to A \to B \to C \to 0$ is an exact sequence of cofinitely generated Λ-modules with C finite, then $\Phi_n(A) \simeq \Phi_n(B)$.*

Proof. Since Φ_n is left exact, we have $0 \to \Phi_n(A) \to \Phi_n(B) \to \Phi_n(C)$. By Lemma 3.3, $\Phi_n(C) = 0$. □

Lemma 3.5. *If $0 \to A \to B \to C \to 0$ is an exact sequence of cofinitely generated Λ-modules with A finite, then $\Phi_n(B) \simeq \Phi_n(C)$.*

Proof. By the snake lemma, taking ϖ^n-torsion gives

$$0 \to A[\varpi^n] \to B[\varpi^n] \to C[\varpi^n] \to A/\varpi^n A.$$

Thus, for some finite H, we have

$$0 \to A[\varpi^n] \to B[\varpi^n] \to C[\varpi^n] \to H \to 0.$$

We split this sequence into two short exact sequences:

$$0 \to A[\varpi^n] \to B[\varpi^n] \to C' \to 0 \qquad (*)$$

and

$$0 \to C' \to C[\varpi^n] \to H \to 0$$

where C' is the image of $B[\varpi^n]$ in $C[\varpi^n]$. Applying Lemmas 3.4 and 3.2 to the second sequence, we have

$$\Phi_n(C') \simeq \Phi_n(C[\varpi^n]) \simeq \Phi_n(C).$$

Again, by the snake lemma, taking Γ_m-invariants of $(*)$ gives

$$0 \to A[\varpi^n]^{\Gamma_m} \to B[\varpi^n]^{\Gamma_m} \to (C')^{\Gamma_m} \to A[\varpi^n]_{\Gamma_m}$$

and thus we have

$$0 \to A[\varpi^n]^{\Gamma_m} \to B[\varpi^n]^{\Gamma_m} \to (C')^{\Gamma_m} \to H'_m \to 0$$

with H'_m finite. In fact, H'_m is finite of size bounded independent of m (as it is a submodule of the finite module A). Splitting up this sequence again, we have

$$0 \to A[\varpi^n]^{\Gamma_m} \to B[\varpi^n]^{\Gamma_m} \to C'_m \to 0$$

and

$$0 \to C'_m \to (C')^{\Gamma_m} \to H'_m \to 0$$

where C'_m is the image of $B[\varpi^n]^{\Gamma_m}$ in $(C')^{\Gamma_m}$.

Taking inverse limits under the trace maps for the first sequence, we obtain

$$0 \to \Phi_n(A) \to \Phi_n(B) \to \varprojlim_m C'_m \to 0.$$

Note that the Mittag-Leffler condition holds for the sequence since A is finite. By Lemma 3.3, $\Phi_n(A) = 0$ and thus $\Phi_n(B) \simeq \varprojlim_m C'_m$.

For the second sequence, taking inverse limits yields

$$0 \to \varprojlim_m C'_m \to \Phi_n(C') \to \varprojlim_m H'_m \to 0.$$

Since C'_m is finite (as it is a finitely generated $(\mathcal{O}/\varpi^n\mathcal{O})[G_m]$-module), the Mittag-Leffler condition holds again. Because the size of H'_m is bounded and the bound is independent of m, the trace maps become multiplication by p. Hence, we have $\varprojlim_m H'_m = 0$ and thus $\varprojlim_m C'_m \simeq \Phi_n(C')$. Hence, $\Phi_n(B) \simeq \Phi_n(C')$ as they are both isomorphic to $\varprojlim_m C'_m$. Since we already have seen $\Phi_n(C') \simeq \Phi_n(C)$, the proof is complete. □

Proposition 3.6. *If A and B are two pseudo-isomorphic cofinitely generated Λ-modules, then $\Phi_n(A) \simeq \Phi_n(B)$.*

Proof. We have an exact sequence of Λ-modules

$$0 \to H \to A \to B \to H' \to 0$$

with H and H' finite. Splitting this sequence into two short exact sequences, we have

$$0 \to H \to A \to A' \to 0$$

and

$$0 \to A' \to B \to H' \to 0$$

with A' the image of A in B. By Lemma 3.5, we have $\Phi_n(A) \simeq \Phi_n(A')$. By Lemma 3.4, we have $\Phi_n(A') \simeq \Phi_n(B)$. Thus, $\Phi_n(A) \simeq \Phi_n(A')$ as desired. □

3.2. Values of the functor.

Proposition 3.7. *If Y is a finitely generated torsion Λ-module with $\mu(Y) = 0$, then $\Phi_n(Y^\vee) = 0$ for all $n \geq 1$.*

Proof. Combining Proposition 3.6 with the structure theorem of finitely generated Λ-modules, it suffices to deal with the case where $Y = \Lambda/f\Lambda$ with $\varpi \nmid f$. By Lemma 3.2, we have

$$\Phi_n(Y^\vee) \simeq \Phi_n(Y^\vee[\varpi^n]) \simeq \Phi_n((Y/\varpi^n Y)^\vee).$$

But $Y/\varpi^n Y \simeq \Lambda/(\varpi^n, f)\Lambda$ which is finite since $\varpi \nmid f$. Thus, by Lemma 3.3, we have $\Phi_n(Y^\vee) = 0$. □

Proposition 3.8. *If $Y \simeq \Lambda/\varpi^t\Lambda$ with $t \geq n$, then $\Phi_n(Y^\vee) \simeq \Lambda/\varpi^n\Lambda$.*

Proof. We compute

$$\Phi_n(\Lambda/\varpi^t\Lambda) = \varprojlim_m \left(\left((\Lambda/\varpi^t\Lambda)^\vee\right)[\varpi^n]\right)^{\Gamma_m} \simeq \varprojlim_m \left((\Lambda/\varpi^n\Lambda)_{\Gamma_m}\right)^\vee \simeq \varprojlim_m \left((\mathcal{O}/\varpi^n\mathcal{O})[G_m]\right)^\vee .$$

Note that the pairing

$$(\mathcal{O}/\varpi^n\mathcal{O})[G_m] \times (\mathcal{O}/\varpi^n\mathcal{O})[G_m] \to (\mathcal{O}/\varpi^n\mathcal{O})$$
$$(\sigma,\tau) \mapsto \begin{cases} 1 & \sigma\tau = 1 \\ 0 & \text{otherwise} \end{cases}$$

induces a Λ-module isomorphism

$$(\mathcal{O}/\varpi^n\mathcal{O})[G_m] \simeq (\mathcal{O}/\varpi^n\mathcal{O})[G_m]^\vee.$$

Let $\pi_{m+1,m}$ be the natural projection $(\mathcal{O}/\varpi^n\mathcal{O})[G_{m+1}] \to (\mathcal{O}/\varpi^n\mathcal{O})[G_m]$. A simple computation shows that the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{O}/\varpi^n\mathcal{O})[G_{m+1}] & \xrightarrow{\simeq} & (\mathcal{O}/\varpi^n\mathcal{O})[G_{m+1}]^\vee \\ \downarrow{\scriptstyle \pi_{m+1,m}} & & \downarrow{\scriptstyle \pi_m^{m+1}} \\ (\mathcal{O}/\varpi^n\mathcal{O})[G_m] & \xrightarrow{\simeq} & (\mathcal{O}/\varpi^n\mathcal{O})[G_m]^\vee \end{array}$$

Thus, we have

$$\Phi_n((\Lambda/\varpi^t\Lambda)^\vee) \simeq \varprojlim_m (\mathcal{O}/\varpi^n\mathcal{O})[G_m]^\vee \simeq \varprojlim_m (\mathcal{O}/\varpi^n\mathcal{O})[G_m] \simeq \Lambda/\varpi^n\Lambda$$

where the first inverse limit is taken under the trace maps and the second inverse limit is taken under the natural projections. □

4. Putting it all together

Proposition 4.1. *We have*

$$\Phi_n(\mathrm{Sel}^{SN^+}(K_\infty, A_f)) = \widehat{\mathrm{Sel}}^{SN^+}(K_\infty, T_{f,n}).$$

Proof. By Lemma 2.1, we have

$$\begin{aligned} \Phi_n(\mathrm{Sel}^{SN^+}(K_\infty, A_f)) &= \varprojlim_m \left(\mathrm{Sel}^{SN^+}(K_\infty, A_f)[\varpi^n]\right)^{\Gamma_m} \\ &\simeq \varprojlim_m \mathrm{Sel}^{SN^+}(K_m, A_{f,n}) \\ &= \widehat{\mathrm{Sel}}^{SN^+}(K_\infty, T_{f,n}). \end{aligned}$$

□

Now we are ready to prove the main theorem of this article.

Proof of Theorem 1.5. By Corollary 2.6, we know that $\mathrm{Sel}^{SN^+}(K_\infty, A_f)^\vee$ is pseudo-isomorphic to

$$\left(\bigoplus_{i=1}^{\#S} \Lambda/\varpi^{t_i}\Lambda\right) \times Y$$

with each $t_i \geq n$ and Y a Λ-torsion module with $\mu(Y) = 0$. By Proposition 3.7, $\Phi_n(Y^\vee) = 0$. By Proposition 3.8, $\Phi_n\left((\Lambda/\varpi^{t_i}\Lambda)^\vee\right) = \Lambda/\varpi^n\Lambda$. Thus, by Proposition 3.6,

$$\Phi_n(\mathrm{Sel}^{SN^+}(K_\infty, A_f)) \simeq (\Lambda/\varpi^n\Lambda)^{\#S}.$$

Proposition 4.1 then completes the proof. □

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