10-6-2009 Tate Thesis Seminar. 10:00 AM.

2 Aims. - compute local zeta fns for finite prime $\mathfrak{p}$.
(easy case)

- introduce the concept of the restricted direct product.

1. Local Computation.

$$\begin{array}{ll} k \supseteq \mathcal{O}_k \ni \pi \quad (\pi)=\mathfrak{p}. & \mathcal{O}_k/(\pi) \cong \mathbb{F}_{p^f} \\ \big|\ \big) \text{ fin.} = n & \quad \big| f \qquad n = e \cdot f. \\ \mathbb{Q}_p \supseteq \mathbb{Z}_p \ni p & \mathbb{Z}_p/(p) \cong \mathbb{F}_p \end{array}$$

ram. index e.

$((\pi) =: \mathfrak{p}.)$

$\xi \in k^+$, a $\mathfrak{p}$-adic variable.

$$\Lambda(\xi) = \lambda(\mathrm{Tr}_{k/\mathbb{Q}_p}(\xi))$$

where $\lambda : \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z}$.

$$x \longmapsto \underbrace{x \pmod{\mathbb{Z}_p}}_{0 \le x < 1}$$

$d\xi$, the additive measure
which makes

$$\int_{\mathcal{O}_k} d\xi = (N\mathfrak{d})^{-1/2}.$$

$\alpha \in k^\times$, a nonzero $\mathfrak{p}$-adic variable.

$\Rightarrow \alpha = \tilde{\alpha} \cdot \pi^\nu$ $\left\{ \begin{array}{l} \pi \text{ a fixed uniformizer of order } 1. \text{ ordinal } 1. \\ \qquad \mathrm{ord}(\pi) = 1. \\ \qquad \text{(normalized)} \\ \nu \in \mathbb{Z}. \\ \tilde{\alpha} \in \mathcal{O}_k^\times = \mathcal{U}, \text{ a unit.} \end{array} \right.$

$$|\alpha| = (Np)^{-\nu}$$
$$= (p^f)^{-\nu}$$

ex $|\pi| = 1/p^f$

$|p| = \left(\frac{1}{p^f}\right)^e = \frac{1}{p^n}.$

$$d\alpha = \frac{Np}{Np-1} \cdot \frac{d\xi}{|\xi|}$$ the multiplicative measure.

so that $\int_{\mathcal{U}} d\alpha = (N\delta)^{-1/2}$.

The equivalence classes of quasi-characters.

Let $c_n(\alpha) : k^\times \to \mathbb{C}^\times$ be a <u>character</u> of $k^\times$
$\searrow S^1$

with conductor $f = p^n$, i.e. $c_n(1+p^n) = 1$.
<u>s.t. $c_n(\pi) = 1$</u>

Then $\{c_n(\alpha)\} \longleftrightarrow \{[\text{quasi-char.}]\}$.

(pf) recall thm. 2.3.1.
The quasi-chars of $k^\times$ are of the form

$$\alpha \longmapsto c(\alpha) = \tilde{c}(\tilde{\alpha}) \cdot |\alpha|^s.$$

determines a class.

any <u>char.</u> of $\mathcal{U}$.

In fact, $c_n : k^\times \to S^1$ is a sort of "trivial ext'n" of
$\cup$
$\tilde{c}_n : \mathcal{U} \to S^1$.

<u>since $c_n(\pi) = 1$.</u>

recall

$\mathfrak{Z}$ = the class of ftns. satisfying

1) $f(\xi)$, $\hat{f}(\xi) \in L^1(k^+) \cap \mathcal{C}(k^+)$.

abs. integrable conti.

2). $f(\alpha)\cdot|\alpha|^{\sigma}$, $\hat{f}(\alpha)\cdot|\alpha|^{\sigma} \in L^1(k^\times)$

for $\forall \sigma > 0$.

(rapidly decreasing $\Rightarrow$ integrability).

rmk. (Schwartz - Bruhat functions)

(Wikipedia) $\downarrow$ conti. dual. (Fourier transform)

(tempered distributions).

on $\mathbb{R}$, $\mathcal{C}^\infty$ and all derivatives are rapidly decreasing.

on torus, just smooth ftn.

cpt already.

on K_v (non-arch),

locally constant ftns of cpt support .

Corresponding Functions of $\mathfrak{Z}$. : (step ftn)

We put

$$f_n(\xi) = \begin{cases} e^{2\pi i \Lambda(\xi)} & \text{for } \xi \in \delta^{-1}\cdot p^{-n} \subseteq k. \\ 0 & \text{for } \xi \notin \delta^{-1}\cdot p^{-n}. \end{cases}$$

Their Fourier transforms.

$$\hat{f}_n(\xi) = \int_{k^+} f_n(\eta)\cdot e^{-2\pi i \Lambda(\xi\eta)}\, d\eta.$$

$$= \int_{\delta^{-1}\cdot p^{-n}} e^{-2\pi i\cdot\Lambda((\xi-1)\cdot\eta)}\, d\eta..$$

$\Rightarrow$ the integral of the additive char. $\eta \mapsto e^{-2\pi i\cdot\Lambda((\xi-1)\cdot\eta)}$

over the cpt. subgp. $\delta^{-1}\cdot p^{-n} \subseteq k^+$.

If $\xi \in 1+\mathfrak{p}^n$, i.e. $\xi \equiv 1 \pmod{\mathfrak{p}^n}$.,
then this character is trivial,

so

$$\int_{\delta^{-1}\cdot\mathfrak{p}^{-n}} d\eta = d\eta(\delta^{-1}\cdot\mathfrak{p}^{-n}).$$

$$= (N(\delta\cdot\mathfrak{p}^n))\cdot d\eta(\mathcal{O}_K)$$

$$= N(\delta)^{\frac{1}{2}}\cdot N(\mathfrak{p})^n.$$

If $\xi \notin 1+\mathfrak{p}^n$, i.e. $\xi \not\equiv 1 \pmod{\mathfrak{p}^n}$,
then the character is <u>nontrivial</u>,

so the integral is zero by cycling elements.

(pf) Let $\xi_0 \in \delta^{-1}\cdot\mathfrak{p}^{-n}$ s.t. $e^{-2\pi i\cdot\Lambda((\xi_0-1)\cdot\eta)}$ ~~$d\eta$~~ $\neq 1$.

$$\int_{\text{cpt gp.}} e^{-2\pi i\,\Lambda((\xi-1)\eta)}\,d\eta \qquad \chi(\xi)$$

$$= \int e^{-2\pi i(\Lambda((\xi-1)\cdot\eta)+\Lambda((\xi_0-1)\cdot\eta))}\,d\eta.$$

$$= \underbrace{e^{-2\pi i\cdot\Lambda((\xi_0-1)\cdot\eta)}}_{\neq 1}\int d\eta.$$

$$\Rightarrow \int d\eta = 0.$$

The ζ-ftns.

Now, we compute the local zeta ftns for such f_n.

cases 1) unramified ($n=0$)
2) ramified
2.1) $n=1$
2.2) $n>1$ "higher ramification".

1). $n=0$.

"conductor $\mathfrak{o}$" nontrivial char. $\rightsquigarrow$ $\exists$ kernel.

The only (unramified) character of type $c_{\mathfrak{o}}$ = the identity char.
(trivial $\Leftrightarrow$ cond $\mathfrak{o}$.) convention.

f_0 : the characteristic ftn. of the set $\mathfrak{d}^{-1}$.

$$\zeta(f_0, \text{id}\cdot|\cdot|^s) = \int_{\mathfrak{d}^{-1}} |\alpha|^s \, d\alpha.$$

Let A_ν be the "annulus" of elements of order ν.
$= \{\xi \in k : |\xi| = (N\mathfrak{p})^{-\nu}\}$.
$\mathfrak{d} = \mathfrak{p}^d$

Then

$$\mathfrak{d}^{-1} = \bigsqcup_{\nu=-d}^{\infty} A_\nu \text{ a disjoint union.}$$

unit $\cdot \pi^\nu$

(π^{-d})

fractional ideal.

$$= \sum_{\nu=-d}^{\infty} \int_{A_\nu} |\alpha|^s \, d\alpha.$$

$$= \sum_{\nu=-d}^{\infty} (N\mathfrak{p})^{-\nu\cdot s} \cdot \int_{\mathcal{O}_k} d\alpha.$$

$$\underset{0<\mathrm{Re}(s)<1}{=} \sum_{\nu=-d}^{\infty} (N\mathfrak{p})^{-\nu s} \cdot (N\mathfrak{d})^{-1/2}.$$

$$= \frac{N\mathfrak{p}^{ds}}{1-N\mathfrak{p}^{-s}} N\mathfrak{d}^{-1/2} = \frac{N\mathfrak{d}^{s-1/2}}{1-N\mathfrak{p}^{-s}}.$$

if $k = \mathbb{Q}_p$,
$$\boxed{\frac{1}{1-p^{-s}}}$$

$\hat{f}_0$ is $NS^{\frac{1}{2}}$ times the characteristic ftn. of $\mathcal{O}$.

$$\zeta(\hat{f}_0, \widehat{Id \cdot | \cdot |^s}) = \zeta(\hat{f}_0, Id \cdot | \cdot |^{1-s}) \qquad \hat{c}(\alpha) = |\alpha| \cdot c^{-1}(\alpha).$$

$$= \underbrace{NS^{\frac{1}{2}}}_{\text{cancel out}} \cdot \int_{\mathcal{O}} |\alpha|^{1-s} \, d\alpha.$$

nu

$$= \sum_{\nu=0}^{\infty} (Np)^{-\nu(1-s)}. \qquad 0 < Re(s) < 1$$

$$|(Np)^{1-s}| < 1.$$

$Re(1-s) < 1$

$0 < Re(s)$

$$= \frac{1}{1-(Np)^{s-1}} = \frac{1}{1-(Np)^{-(1-s)}} \qquad \frac{1}{1-p^{(1-s)}}$$

//.

2.) ramified case $(n > 0)$

$$\zeta(\underset{\sim}{f_n}, c_n \cdot | \, |^s) = \int_{\delta^{-1} \cdot p^{-n}} e^{2\pi i \Lambda(\alpha)} \cdot c_n(\alpha) \cdot |\alpha|^s d\alpha.$$

defn

$$= \sum_{\nu=-d-n}^{\infty} \left(Np^{-\nu s} \int_{A_\nu} e^{2\pi i \Lambda(\alpha)} \cdot c_n(\alpha) \, d\alpha \right)$$

$$\vdots \qquad \Rightarrow \boxed{\nu = -d-n \text{ 만 산다}}.$$

Then, of course, $\zeta(\hat{f}, \hat{c})$ not identically zero.

Local f.e. says

$$\rho(c) = \frac{\zeta(f, c)}{\zeta(\hat{f}, \hat{c})} = \boxed{\frac{1-p^{s-1}}{1-p^{-s}}} \qquad \text{bad primes?}$$

: meromorphic ftn on $\mathbb{C}$.

RESTRICTED.

2. Abstract Direct Product.

2.1. Intro. - construction. and basic top. properties.
2.2. Characters
2.3. Measures

2.1.

Aim. - construction
- It is also LCAG.

Underlying Set and Alg. Str. - easy ; top - not-so-easy.

$\{p\}$: the set of countably (infinite) indices.
(varying over finite & infinite primes in applications to arithmetic)

Want to assign
$$p \rightsquigarrow G_p \text{, a LCAG.}$$
with condition
for $\forall\forall p \in \{p\}$,
$$p \rightsquigarrow \underbrace{H_p}_{\text{fixed. for } \forall p.} \subseteq G_p \text{ open \& cpt. subgp}$$

$\Downarrow$

form a new abstract gp G.

$G \ni \alpha = (\cdots, \alpha_p, \cdots)$: an element of G.
= an infinite vector of $\prod_p G_p$.

with the condition
$$\alpha_p \in H_p \quad \text{for } \forall\forall p \in \{p\}.$$

multiplication of G : defined componentwisely.
(multiplicative notation)

Why don't we use just the direct product?

- "closer to global"
  cheating : $\mathbb{Q} \overset{!}{\hookrightarrow} \mathbb{A}_{\mathbb{Q}} = \prod_p' \mathbb{Q}_p \times \mathbb{R} \subseteq \underbrace{\prod_p \mathbb{Q}_p}_{\text{too big}} \times \mathbb{R}.$

(S also control class#)
S-class gp $\rightsquigarrow$ 1.
class gp. invert
representative

diagonal embedding

($\because$ finite factors of denominators)
$\hookrightarrow$ fin. # of poles (ftn. field case)

too big
to reflect the global property
the property of $\mathbb{Q}$.

- (topologically) LCAG (We will prove this)

Now we topologize G !

Fix a finite subset $S \subseteq \{p\}$.
including $\forall p$'s for which $\underset{\text{open cpt subgp}}{H_p}$ is not defined.

(That means,
for those $p \in S$, G_p only occurs.

Define $G_S = \{ \alpha \in G : \alpha_p \in H_p \text{ for } \underbrace{\forall p \notin S}_{\text{almost all}} \}$

[$\Lambda(D)$ ftn. field case: pole $\doteqdot$ $\{2\}$] (of course, $\alpha_p \in G_p$ (for $p \in S$).)

Then we have a natural isom. of top. gps. (by topologizing G_S).

$$G_S \cong \underbrace{\prod_{p \in S} G_p \times \prod_{p \notin S} H_p}_{\text{direct product of l.c. gp.}}.$$

$\leftarrow$ gives the usual product topology

$\prod_{p \notin S} H_p$: cpt. by Tychonoff Thm. (use the cptness of H_p)

$\rightsquigarrow$ (finite product of LC spaces) $\times$ (cpt sp.)
$\Rightarrow$ locally cpt. in the product topology.

[To define a topology on a group G
it suffices to give a fundamental system of nbhds of 1 in G.

To topologize G,

take a fundamental system of nbhds of 1 in G.
as the set of nbhds of 1 in G_S !

Question. This topology depends on S ?
the set of indices

Answer. No! (Fortunately?)

We explain this.

lemma. The set of all "parallotopes" of the form

$$N = \prod_p N_p \quad (\subseteq G.)$$

where N_p = a nbhd of 1 in G_p $\exists^\forall p$,
with
$N_p = H_p$ for $^{\forall\forall} p$..

~~rmk. We use~~
also gives a fundamental system of nbhds of 1 in G.

(rmk. We use the condition
$H_p \underset{\text{open}}{\subseteq} G_p$ here.)

$\Downarrow$

This ~~product~~
top is not prod top
use H_p
not G_p.

cpf. The def'n of prod. top. of G_S implies.

$\forall$ nbhd of 1 in G_S contains a parallotope. $N = \prod_p N_p$

$\Rightarrow \{N\}$ is finer than $\{$nbhds of 1 in $G_S\}$.

On the other hand,
since $N_p = H_p$ for $^{\forall\forall} p$,

$$\left(\prod_p N_p\right) \cap G_S = \prod_{p \in S} N_p \times \prod_{p \notin S} (N_p \cap H_p)$$

is a nbhd of 1 in G_S. ($\Rightarrow$ a nbhd of 1 in $G_S \subseteq N$)

$\Rightarrow \{$nbhds of 1 in $G_S\}$ is finer than $\{N\}$.

$\square$.

<u>observation</u>.

- G_S is open in G. ($\Leftarrow$ G_S itself is a parallotope.)
- two topologies on G_S
  - the subsp. top. induced from G.
  - the prod. top define!!

  coincide.

  ($\Leftarrow$ obvious since ~~~~).

not prod top

- <u>bonus</u>.

  a cpt nbhd of 1 in G_S

  is a cpt nbhd of 1 in G.

$\Rightarrow$ G is locally compact.

Summarizing what we've constructed.

def. The LCAG G is called the restricted direct product
of the gp. G_p (LCAG.)
relative to the subgps H_p (open, cpt)

Now we try to investigate cpt. nbhds explicitly.

convention on identification.

$$\imath_p : G_p \hookrightarrow G \qquad \text{NOT DIAGONAL HERE!}$$

$$a_p \longmapsto (1, \cdots, 1, \underset{\substack{\uparrow \\ p\text{-th slot.}}}{a_p}, 1, \cdots 1)$$

$$G_p \underset{\uparrow}{\simeq} \imath_p(G_p) \subseteq G.$$

natural isom. of top. gps. (alg. & top.)

[the inverse of open is open!
. open $\to$ open!

Since the cpnts a_p of any element a of G
lie in H_p for $\forall\forall p.$,

$a \in G_S$ for some S,

i.e. $G = \bigcup_S G_S$.

That means Study of G
$\downarrow$ reduces to.
Study of G_S //.