10-20-2009 Tate Thesis Seminar 10:00AM.

The abstract direct product. - cont'd.

Recall the underlying set of G.

$\{p\}$: the set of indices.

G_p : a LCAG.

H_p : open and cpt subgp. of G_p.

$G \ni \alpha = (\cdots, \alpha_p, \cdots)$: an element of G

= an infinite vector of $\prod_p G_p$.
with the condition
$\alpha_p \in H_p$ for $^{\forall\forall} p \in \{p\}$.

Gp. str. : componentwisely
(We use multiplicative notation)

Now we topologize G!

Fix a finite subset $S \subseteq \{p\}$ ~~including ...~~
including $^{\forall}p$'s for which H_p (open, cpt) is NOT defined.

(That means,
for those $p \in S$, G_p only occurs.)

Define a (abstract) subgp. $G_S = \{\alpha \in G : \alpha_p \in H_p$ for $^{\forall}p \notin S\}$ (almost all!)

(of course, $\alpha_p \in G_p$ always).

(Geometric interpretation)
If we regard G as meromorphic fns on a curve,
G_S can be understood as meromorphic fns with prescribed poles (divisor) S.

(We do not use the concept of modulus here) → (If we deal with adeles of fn. fields, we can specify the multiplicity by valuation.) only deg 1 here.

Now we topologize G_S first. using the following natural isom.

$$G_S \cong \prod_{p \in S} G_p \times \prod_{p \notin S} H_p .$$

give top. ← Here, we gave the usual product topology.
i.e. the direct product of LCAG.

Moreover, $\prod_{p \notin S} H_p$ is compact by Tychonoff thm.
(We use the cptness of H_p here)

Then (finite product of LC spaces) × (compact sp)
→ locally compact in the product topology.

Note. ~~Since G is a groups it suffices~~

To define a topology on a group G,
it suffices to give a fundamental system of nbhds of 1 in G.

Now we define a topology of G as follows.
: Take a fundamental system of nbhds of 1 in G
as the set of nbhds of 1 in G_S.

(natural)
Question. This topology depends on the choice of S ?

Answer No! (fortunately).

We explain this.

The following lemma gives an "S-free" fundamental system of nbhds of 1 in G.

<u>lemma</u>. The set of all "parallotopes" of the form

$$N = \prod_p N_p \quad (\subseteq G)$$

where N_p = a nbhd of 1 in G_p <u>for $^\forall p$.</u>

with

$N_p = H_p$ <u>for almost all p</u>. (Note that <u>H_p is also open!</u>)
use here!

also gives a fundamental system of nbhds of 1 in G.

(pf) It suffices to say

① $\{N\}$ is finer than $\{$nbhds of 1 in $G_S\}$.

② coarser.

①

The def'n of product top. of G_S

⇓ implies

$^\forall$ nbhd of 1 in G_S contains a parallotope $N = \prod_p N_p$.

$\Rightarrow$ $\{N\}$ is finer than $\{$nbhds of 1 in $G_S\}$.

② On the other hand,

since $N_p = H_p$ <u>for almost all p</u>,

$$\left(\prod_p N_p\right) \cap G_S = \prod_{p \in S} N_p \times \underbrace{\prod_{p \notin S} (N_p \cap H_p)}_{\text{most of them } = H_p} \quad \left(\subseteq \prod_p N_p.\right)$$

is a nbhd of 1 in G_S. (as prod top, ok)

$\Rightarrow$ $\{$nbhds of 1 in $G_S\}$ is finer than $\{N\}$

☒

<u>Observations</u>.

- The topology of G is <u>not</u> the product topology, finer.
- G_S is open in G ($\Leftarrow$ G_S itself is a parallotope.)
- By construction,

  the product top of G_S = the subsp. top. of G_S induced from G.

- Then Why do we use S? → Control the poles S ↔ Riemann-Roch problems.
  → much <u>more</u> <u>explicit</u>

· bonus.

· a compact nbhd of 1 in G_S is a compact nbhd of 1 in G.

$\Rightarrow$ G is locally compact. ($\Rightarrow \exists$ Haar measure, ...) good!!

Summarizing what we've constructed,

def. The LCAG G is called the restricted direct product of the gps G_p (LCAG) relative to the subgps H_p (open and cpt.)

Now we try to investigate cpt. nbhds explicitly.

Convention (on identification)

$i_p : G_p \hookrightarrow G$ NOT DIAGONAL HERE!

$\alpha_p \longmapsto (1, \cdots, 1, \alpha_p, 1, \cdots 1)$

↑ p-th slot.

Then

$G_p \simeq i_p(G_p) \subseteq G$.

↑ natural isom. of top. gps. (algebraic (ok) and topological)

i_p is conti. : if $V \subseteq G$ open, (espeicially, a nbhd of 1)

$i_p^{-1}(V) := V_p$ is open. (a nbhd of 1_p)

by parallotope description.

i_p is open : Let $\mathcal{U} \subseteq G_p$ an open subset (especially, a nbhd of 1)

Then $i_p(\mathcal{U}) = \underset{\sim}{V} \cap G_p$

for some $V \underset{\text{open}}{\subseteq} G$ open with $(i_p(\mathcal{U}))_p = V_p$.

recipe of V : example (Take V : $i_p(\mathcal{U}) \times \prod_{q \neq p} G_q$)

$\Rightarrow$ i_p is open with the subspace top. of $i_p(G_p)$.

Covering Property.

Since the components α_p of any element α of G lie in H_p for almost all p,

$\alpha \in G_S$ for some finite subset $S \subseteq \{p\}$.

$\Rightarrow G = \bigcup_S G_S$.

That means ...

Study of G

↓ reduces to

Study of G_S's (much easier, set-theoretically explicit; just the product topology)

↓ one more step.

Consider the (useful) subgp. $G^S \subseteq G_S$ "more easier" : even compact !! (useful when dealing with measures.)

$G^S = \{ \alpha \in G_S : \alpha_p = 1 \text{ for } \forall p \in S \}$ ("deprived", "killed at S")

of course, $\alpha_p \in H_p$ for $\forall p \notin S$.

$\Rightarrow G^S \simeq \prod_{p \notin S} H_p$, so a compact subgp of G_S.

(H_p, cpt + Tychonoff thm.)

$\Rightarrow G_S \simeq \underbrace{\prod_{p \in S} G_p}_{\text{finite product of } G_p} \times \underbrace{G^S}_{\text{cpt.}}$

$\subseteq G$.

Now, we describe compact nbhds of G.

<u>lemma 3.12.</u>

$C \subseteq G$ is <u>relatively compact</u>, i.e. has a cpt. closure.

$\Longleftrightarrow$ $C \subseteq \prod_p B_p$, a parallelotope,

where $B_p \subseteq G_p$, a cpt subset, <u>for $\forall p$</u>

with $B_p = H_p$ <u>for almost all p</u>.

(pf) ($\Rightarrow$)

<u>claim.</u> (Any compact subset of G) $\subseteq G_S$ for some S.
$=: C$

(pf of claim)

We know $\underset{\underset{C}{\text{Ul cpt.}}}{G} = \bigcup_S G_S$ an open covering of G.

Since C is cpt.,

$$C \subseteq \bigcup_{S_i} G_{S_i} \quad \text{for } i = 1, \cdots, n.$$

Since $G_S \cup G_T \subseteq G_{S \cup T}$,

$$\bigcup_{\text{fin.}} G_S \subseteq G_{\bigcup_{\text{fin.}} S}.$$

$\Rightarrow$ $C \subseteq G_S$ for some S; the claim is proved. //

Any compact subset of G_S is of the form

$$C = \prod_{p \in S} C_p \times \prod_{p \notin S} C_p'$$

where C_p is a cpt. subset of G_p.

C_p' is a cpt subset of H_p.

$\Rightarrow$ $\exists$ a parallotope $\prod_p B_p$ containing C.

(⇐) Any parallotope $\prod_p B_p$ is obviously a cpt. subset of G_S for some S, thus of G.

$$C \subseteq \prod_p B_p \subseteq G_S \subseteq G$$

$\Rightarrow$ C has a cpt nbhd.

$\square$

Before moving on characters, we examine the word "parallotope".

Lattices vs. Parallotopes (in adéles).

We consider "the" explicit example : $\mathbb{A}_{\mathbb{Q}} = \prod_p{}' \mathbb{Q}_p$

where p runs over finite and infinite primes.

Then we have

$\Delta : \mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$: the diagonal embedding.

$a \longmapsto (a, \cdots, a, \cdots)$

($\mathbb{R}$ ← no integral str. here!)

Let S be a finite subset of primes. (including ∞ prime)

$\mathbb{A}_S := \{ \alpha \in \mathbb{A}_{\mathbb{Q}} : \alpha_p \in \mathbb{Q}_p$ for $\forall p \in S$, $\alpha_p \in \mathbb{Z}_p$ for $\forall p \notin S \}$. a parallotope.

Since $\mathbb{Q}_\infty$ ($\mathbb{R}$ or $\mathbb{C}$) has no integral str, we always include ∞ into S.

Then $\mathbb{A}_{\mathbb{Q}} = \bigcup_S \mathbb{A}_S$.

//.

$[0,1]$ ($(0,1)$)

$\cap$

$\mathbb{Z} \subseteq \mathbb{Q} \hookrightarrow \mathbb{R}$.

$\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, $\mathbb{Q}_p \supseteq \mathbb{Z}_p$.

$\mathbb{Q}_p / \mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z}$.

In this sense, we may consider $\mathbb{Z}_p$ as a lattice of $\mathbb{Q}_p$.

different point of views

Fact. $\mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}}$ is compact. $\Rightarrow$ $\mathbb{Q}$ is regarded as a lattice of $\mathbb{A}_{\mathbb{Q}}$.

$\Rightarrow$ $\mathbb{Q}$ can be regarded as a lattice of $\mathbb{Q}_p$

Characters. (and Quasicharacters)

We (quasi-) characterize (quasi-) characters. =).

Let $c(\alpha)$ be a quasi-character of G,
i.e.

$$c : G \longrightarrow \mathbb{C}^{\times}$$ is a conti. multiplicative mapping. (gp. hom.)

Let c_p be the restriction of c to G_p,
i.e.

$$c_p : G|_p \underset{\uparrow}{=} G_p \longrightarrow \mathbb{C}^{\times}$$

our convention

$$c_p(a_p) = c(\alpha_p) = c(1, 1, \ldots, 1, \underset{p\text{-th slot}}{\alpha_p}, 1, \ldots, 1)$$

$\Rightarrow$ c_p is also a quasicharacter.

The roles of next 2 lemmas.

- give iff-condition on "which collection of $\{c_p\}_p$ of quasicharacters of $\{G_p\}_p$ come from the ones of G? (by restriction)."
- give bijection on the duality.

lemma 3.2.1. (necessary condition).

Let $c : G \to \mathbb{C}^{\times}$ be a quasicharacter of G
$c_p : G_p \to \mathbb{C}^{\times}$ be its restriction to p-cpnt.

Then

- c_p is trivial on H_p for almost all p.
- For any $\alpha \in G$,

$$c(\alpha) = \prod_p c_p(\alpha_p)$$

(The first statement implies this product is finite)

(pf)

(pf) Let $\mathcal{U} \subseteq \mathbb{C}^\times$, a nbhd of 1 in $\mathbb{C}^\times$
containing no multiplicative subgp. except $\{1\}$.

Let $N = \prod_p N_p$ be a nbhd of 1 in G
such that $c(N) \subseteq \mathcal{U}$.

(recipe of N
: $c^{-1}(\mathcal{U}) \cap$ (any nbhd of 1 in G))

Select, a finite subset S such that S contains all p for which $N_p \neq H_p$.
(That means $p \notin S \Rightarrow N_p = H_p$).

Then

$$G^S = \{ \alpha \in G_S : \alpha_p = 1 \text{ for } p \in S \}$$
$\alpha_p \in H_p$ for $\forall p$

$\subseteq N$ — implies.

$\Rightarrow c(G^S) \subseteq \mathcal{U}$.

$\Rightarrow c(G^S) = 1$.

$\Rightarrow c(H_p) = 1$ for $p \notin S$. //

Now we prove the product description.

Let α be a (fixed) elt. of G.

(We impose on S the further condition
$\alpha \in G_S$ for some S).

We enlarge $S \subsetneq S'$ such that
$\alpha \in G_{S'}$

Then we can write

$$\alpha = \prod_{p \in S'} \alpha_p \cdot \alpha^{S'}$$, where $\alpha^{S'} \in G^{S'}$.

$$\Rightarrow \; c(\mathfrak{a}) \overset{\checkmark}{=} \prod_{p \in S'} c(\mathfrak{a}_p) \cdot c(\mathfrak{a}^{S'})$$ (c is multiplicative!)

$$= \prod_{p \in S'} c(\mathfrak{a}_p)$$

since $c_p(\mathfrak{a}_p) = 1$ for $p \notin S$
and
$$p \notin S' \Rightarrow p \notin S .$$

$\square$

lemma 3.2.2. (Sufficient Condition)

Let c_p be a quasicharacter of G_p for each p,
with c_p is trivial on H_p. for almost all p.
If we define
$$c(\mathfrak{a}) \overset{def}{:=} \prod_p c_p(\mathfrak{a}_p) ,$$
then we obtain a quasicharacter of G.

(pf) $c(\mathfrak{a})$ is obviously multiplicative.

c is continuous?

Select S containing all p for which $c_p(H_p) \neq 1$.
(i.e. if $p \notin S$, $c_p(H_p) = 1$)

Let $s = \#(S)$.

A nbhd $\mathcal{U}$ of $1 \subseteq \mathbb{C}^\times$ is given.

Choose a smaller nbhd V such that $V^s \subseteq \mathcal{U}$

$\quad V^s = \{ v_1 \cdots v_s \in \mathcal{U} : v_i \in V \text{ for } \forall i \}$. $\quad \mathcal{U} \subseteq \mathbb{C}^\times$

(clue for recipe of V) $\to$ (take " $\sqrt[s]{\mathcal{U}}$ ")

Let N_p be a nbhd of 1 in G_p
such that
- $c_p(N_p) \subseteq V$ for $\forall p \in S$.
- $N_p = H_p$ for $\forall p \notin S$.

Then

$$c\left(\prod_p N_p\right) \subseteq V^s \subseteq \mathcal{U}.$$

(This is nothing but "the ε-δ argument"

"ε-nbhd $= \mathcal{U}$.

δ-nbhd $= \prod_p N_p$.") (*) see the last page.

c is continuous ☒.

Quasi-characters ⇝ Characters.

↓

We will analyze the dual of G.
(Pontryagin).

<u>Observation</u>.

$c(\mathfrak{a}) = \prod_p c_p(\mathfrak{a}_p)$ is a character

$\Leftrightarrow$ $^\forall c_p$ is a character. ← study these guys.

Let $\widehat{G_p} = \mathrm{Hom}_{\text{conti.}}(G_p, S^1)$, $S^1 \subseteq \mathbb{C}^\times$.

$H_p^* = H_p^\perp = \mathrm{Ann}(H_p)$

$= \{ c_p \in \widehat{G_p} : c_p(\mathfrak{a}_p) = 1 \text{ for } ^\forall \mathfrak{a}_p \in H_p \}.$

Now we investigate their topological properties.

H_p is compact $\Longrightarrow$ $\widehat{H_p} \simeq \widehat{G_p}/H_p^\perp$ is discrete.
(↑ Pontryagin duality)

$\Rightarrow$ $H_p^\perp$ is open.

open ↔ open ↙ discrete, {id} is open.

(since $H_p^\perp = \pi^{-1}(id)$ ⇒ open

with $\pi : \widehat{G_p} \longrightarrow \widehat{G_p}/H_p^\perp$)
$\qquad\qquad\qquad\qquad\qquad \ni id$

(w/o Pontryagin duality, consider

$$H_p^{\perp} = W(H_p, U) = \{ \zeta_p \in \widehat{G_p} : \zeta_p(H_p) \subseteq U \}.$$

where $U \subseteq S^1$ ← small enough to contain no nontrivial subgp of S^1

$\Rightarrow H_p^{\perp}$ is open by cpt-open topology.).

H_p is open. $\Rightarrow$ (By the def'n of the quotient topology.) G_p / H_p is discrete

$\Rightarrow$ (Pontryagin duality) $\widehat{G_p / H_p} \simeq H_p^{\perp}$ is cpt.

$\therefore H_p^{\perp}$ is open and cpt in $\widehat{G_p}$.

We expect, here, when we define $\widehat{G}$, $H_p^{\perp}$ will play the same role as H_p.

Thm 3.2.1. (Duality of restricted direct product)

$\left(\prod_p{}' \widehat{G}_p \quad \text{relative to } H_p^{\perp} \right)$ $\qquad \ni \prod_p c_p(\alpha_p)$

$\cong$ as topological groups. (canonical) $\qquad \updownarrow$

$\widehat{G} = \mathrm{Hom}_{\text{conti.}}(G, S^1)$. $\qquad \ni c(\alpha)$

(rmks. local decomposition of global characters ...)

(pf) The (natural) map is defined by (identification)

$\widehat{G} \ni c \overset{!}{=} (\cdots, c_p, \cdots) \in \prod_p \widehat{G}_p$

with $c(\alpha) = \prod_p c_p(\alpha_p)$.

Algebraic Isomorphism

Already proved by the above 2 lemmas + (char. $\Leftrightarrow$ char.)

Topological Argument

$c = (\cdots, c_p, \cdots) \longrightarrow Id$ as a character of G.

$\Longleftrightarrow$ $c(B) \longrightarrow 1$ for a large compact $B \subseteq G$.

$\Longleftrightarrow$ $c(\prod_p B_p) \longrightarrow 1$ for $B_p \underset{\text{cpt}}{\subseteq} G_p$ for $\forall p$.

($B_p = H_p$ for $\forall\forall p$).

$\Longleftrightarrow$ $c_p(B_p) \longrightarrow 1$ wherever $B_p \neq H_p$.

and

$c_p(B_p) = c_p(H_p) = 1$ at the remaining p.

(Since H_p is a <u>subgroup</u>, $c_p(H_p)$ can be close 1 only if $c_p(H_p) = 1$).

$\Longleftrightarrow$ $\varsigma_p \longrightarrow 1 = \mathrm{Id}$ as a character of G_p
for a finite # of p
($B_p \neq H_p$ case)

and

$\varsigma_p \in H_p^{\perp}$ at the other p.

$\Longleftrightarrow$ $C \longrightarrow \mathrm{Id}$ ~~as a character of~~
in the restricted product of G_p.

$\square$

Fact. X, Y : top. sp. (Munkres, pg 104)

$f: X \to Y$ is conti.

$\Leftrightarrow$ For any $x \in X$ and any nbhd V of $f(x)$,

there is a nbhd U of x

such that $f(U) \subseteq V$.

(pf) ($\Rightarrow$)

If we take $U = f^{-1}(V)$, $f \cdot f^{-1}(U) \subseteq V$ //

($\Leftarrow$) Let $V \subseteq Y$ be an open subset of Y.

x be a point of $f^{-1}(V)$.

Then

$f(x) \in V$.

By assumption,

$\exists$ a nbhd U_x of x such that $f(U_x) \subseteq V$.

$\Rightarrow$ $U_x \subseteq f^{-1}(V)$.

$\Rightarrow$ ~~$\{U_x\}_{x \in f^{-1}(V)} = f^{-1}$~~

$\bigcup_{x \in f^{-1}(V)} U_x = f^{-1}(V)$, so open

$\boxtimes$