9-15-2009 Kalin Tate Thesis 10:00 AM.

"Big" Intro. to Tate thesis.

Five Easy Pieces.

1. L-functions.
2. Cribnotes on Tate's thesis.
3. Merom. conti. & f.e. for $\zeta(s)$
4. Harmonic Analysis in NT.
5. idéles and adéles.

a) ~~Gelfan~~
Gelbart's Lecture @ Shandong U (online)
b) Knapp's article in Notices
c) Course Note on Tate's thesis (Buzzard, K. Conrad)
d) Other course notes (Kedlaya for 5.)
e) Ramakrishna - Valenza.

1) Examples of arithmetic info in L-ftns.

a) $\zeta(-11) = \dfrac{691}{32760} \Rightarrow 691 \mid h(\cancel{\ \ }\ \mathbb{Q}(\mu_{691}))$

b) ACNF.

$$\lim_{s \to 1} (s-1) \cdot \zeta_K(s) = \frac{2^{r_1+r_2} \pi^{r_2} \cdot h_K \cdot R_K}{|W_K| \cdot \sqrt{|d(K)|}} \quad (*).$$

$\to \cdot \quad \cdot \leftarrow$
0 1

$s \longleftrightarrow 1-s$

(*) + f.e.

$\zeta_K(s)$ near $s=0$ looks like $s^{\overset{\text{rank of } \mathcal{O}_K^\times}{\downarrow}}{}^{r_1+r_2-1} \cdot \left(\dfrac{-h_K \cdot R_K}{|W_K|} \right) + \cdots$

Change K for an E/K

(CNF) $\rightsquigarrow$ (BSD).

Thm (Hecke, 1916)

The function

$$L(s,\chi) = \prod_{p} \left(1 - \frac{\chi(p)}{N(p)^s}\right)^{-1}$$

has merom. conti. and f.e.

$$A^s \cdot \Gamma^{r_1}\left(\tfrac{s}{2}, \chi\right) \cdot L(s,\chi) = W(\chi) \cdot A^{1-s} \cdot \Gamma^{r_1}\left(\tfrac{1-s}{2}, \bar{\chi}\right).$$

$\Gamma^{r_2}(s,\chi)$ $\qquad$ $L(1-s, \bar{\chi})$ $\qquad$ $\Gamma^{r_2}\left(\tfrac{1-s}{2}, \bar{\chi}\right)$

Thm (Tate, 1950)

We have

$$\zeta(f,c) = \zeta(\hat{f}, c^{\vee})$$

where $\zeta(f,c) = \int_{\mathbb{I}_K} f(a)\, c(a) \cdot \underbrace{d^{\times}a}_{\text{mult. Haar measure}}.$

$\hat{f}$ is the Fourier transform of $f \in \mathcal{S}(\mathbb{I}_K, \mathbb{C})$?

$c : \mathbb{I}_K \xrightarrow{\text{conti.}} \mathbb{C}^{\times}$ which is trivial on $K^{\times}$.

Example:

$$f = (f_v)_v$$

$$f_\infty = e^{-\pi x^2}, \quad f_p(x) = 1_{\mathbb{Z}_p}$$ characteristic fn.

→ good test fn.

$$c(a) = |a|_{\mathbb{A}}^{s}$$

In this setting,

$$\zeta(f, c) = \underbrace{\pi^{-s/2} \cdot \Gamma(s/2) \cdot \zeta(s)}_{\xi(s)}$$

$$\int_{\mathbb{I}_K} f(a) \cdot c(a)\, d^\times a = \int_{\mathbb{Q}^\times \backslash \mathbb{I}_K} |a|_{\mathbb{A}}^{s} \cdot \sum_{q \in \mathbb{Q}^\times} f(qa)\, d^\times a.$$

$$= \int_{(0,\infty) \times \hat{\mathbb{Z}}^\times} |a|_{\mathbb{A}}^{s} \cdot \sum_{n \in \mathbb{Z} \setminus \{0\}} f(na)\, d^\times a.$$

$$f(na_p) \equiv 1.$$

∞-part

$$\int_0^\infty \cancel{|a|_\infty^s}\, x^s \cdot \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{-\pi n^2 x^2}\, d^\times x.$$

and this is exactly the integral

for $\xi(s) = \pi^{-s/2} \cdot \Gamma(s/2) \cdot \Gamma(s)$

3. merom. conti. and f.e. for $\xi(s)$.

Ingredients for 3)

- Clever def'n for $\xi(s)$.

$$\xi(s) = \underline{\pi^{-s/2} \cdot \Gamma(s/2)} \cdot \Gamma(s)$$

$$\Rightarrow \xi(s) = \xi(1-s)$$

- Two ways to evaluate $\xi(s)$.
  - a) brute force.
  - b) Mellin transform of θ-ftn.

$$\xi(s) = \int_0^\infty (\theta(t) - \underset{\text{pole}}{\underbrace{1}}) \, t^{s-1} \, dt$$

use $t \cdot \theta(t) = \theta(1/t)$ (Poisson Summation)

$$\xi(s) = \int_1^\infty (\theta(t) - 1)(t^{s-1} + t^{-s}) \, dt.$$

$$\underline{- \frac{1}{1-s} - \frac{1}{s}}.$$

symmetric under $s \longleftrightarrow 1-s$