9-22-2009 Ander Tate's Thesis 10:00 AM.

- Review of F.A. on LCAG.

Let G be a LCAG.

$\hat{G}$, the gp. of chars on G (into $S^1 \subseteq \mathbb{C}^\times$)

$\chi \in \mathrm{Hom}_{\text{conti.}}(G, S^1)$

$|\chi(g)| = 1 \quad \forall g \in G.$

<u>Facts</u> (Rudin, Fourier Analysis on Groups.)

1) $\hat{G}$ is also a LCAG.

Tate $G \simeq \hat{G}$ identify?

$\mathbb{R} \simeq \hat{\mathbb{R}}$.

$G \times \hat{G} \longrightarrow S^1 \subseteq \mathbb{C}^\times$

conti. 가 되게 하는 top. 준다.

Our G's will be explicitly identified with $\hat{G}$.

2) G and $\hat{G}$ have Haar measures. (translation invariants)

compatible with Fourier transforms

3). closed subgps of G $\longleftrightarrow$ quotient of $\hat{G}$ by annihilator.

H

$\mathrm{Ann}(H) = H^\perp = \Lambda$.

$\hat{H} \simeq \hat{G}/H^\perp$.

$\to$ Can choose μ, $\hat{\mu}$ so that $f \in L^1(G, \mathbb{C}) \cap C^0(G, \mathbb{C})$

$$\hat{f}(\chi) := \int_G f(g) \underbrace{\langle -g, \chi \rangle}_{\chi(-g)} \, d\mu$$

<u>Thm.</u> (Fourier Inversion Formula)

$$f(g) = \int_{\hat{G}} \hat{f}(\chi) \langle g, \chi \rangle \, d\hat{\mu}$$

$$= \hat{\hat{f}}(-g)$$

(~~where~~ if $f \in L^1(G)$, $\hat{f} \in L^1(\hat{G})$)

conti.

(몇몇 pt 바꿔도...)

<u>Local Theory</u>

Let K be a cpltion of some # field w.r.t. p.

i.e.

$$K = \begin{cases} \mathbb{R} & \text{if } p \text{ real,} \\ \mathbb{C} & \text{if } p \text{ cpx.} \\ \text{finite ext'n of } \mathbb{Q}_p & \text{if } p \text{ fin.} \end{cases}$$

Our first order business is $G = K^+$.

Thm. The character group $\widehat{K^+}$ is isomorphic to K^+.

Lemma. If χ is an nontrivial character of K^+, then

$$\eta \longleftrightarrow (\xi \longmapsto \chi(\eta\cdot\xi))$$

correspondence is an isomorphism of top. gps.

$$K^+ \xrightarrow{\simeq} \widehat{K^+}.$$

(pf) (i) $\chi(\eta\cdot(\ ))$ is a character. ✓

(ii) $\chi((\eta_1+\eta_2)\cdot\xi) = \chi(\eta_1\xi)\cdot\chi(\eta_2\xi)$ conti. gp. hom. ✓

(iii) If $\chi(\eta\cdot\xi) = 1 \ \forall \xi \in K^+$, then $\eta\cdot K^+ \neq K^+$ $\Rightarrow \eta = 0$.) injective.

(iv). If $\chi(\eta\cdot\xi) = 1 \ \forall \eta \in K^+$.
$\Rightarrow \xi K^+ \neq K^+$
$\Rightarrow \xi = 0$.

So the img of $\eta \longmapsto \chi(\eta -)$ is dense in $\widehat{K^+}$.

(証)

Dense + top. argument $\Rightarrow$ closed $\Rightarrow$ surjective. 証

Now we produce a non-trivial character.

Let p be the prime below $\mathfrak{p}$.
any place of $\mathbb{Q}$.

Write R for the cpltion of $\mathbb{Q}$ w.r.t p.

$$\begin{array}{c} K \\ | \\ R. \end{array}$$

We will define $\lambda : R \to (\mathbb{R} \bmod 1)$.

If $R = \mathbb{R}$,

$$\lambda(x) = -x \pmod 1$$

If $R = \mathbb{Q}_p$ (p, finite),

For $x \in \mathbb{Q}_p$,

to be ν s.t. $p^\nu \cdot x \in \mathbb{Z}_p$.

Take $n \in \mathbb{Z}$

s.t. $n \equiv p^\nu \cdot x \bmod p^\nu$.

$\lambda(x) = n/p^\nu$. the tale of p-adic #. mod p-adic int.

Define

$$\Lambda : K \to (\mathbb{R} \bmod 1)$$

$$\Lambda(\xi) = \lambda(\mathrm{Tr}_{K/R}\, \xi).$$

Glenn.
$\mathbb{Q}$에서 한다음
continuity 이용
→ nontrivial

$$\left(\xi \mapsto e^{2\pi i \cdot \Lambda(\xi)}\right) \in \widehat{K^+}$$

Our isomorphism is

$$\eta \longmapsto \left(\xi \longmapsto e^{2\pi i \Lambda(\eta\cdot\xi)} \right).$$

Now, we figure out Haar measures.

Haar Measure

Since K^+ is a LCAG, we have a Haar measure.

We pick

$$d\xi = \mu = \begin{cases} \text{ordinary Lebesgue if } K = \mathbb{R}. \\ 2dxdy \quad 2\cdot\text{ordinary Lebesgue if } K = \mathbb{C}. \\ \text{the measure that gives} \\ \mu(\mathcal{O}_K) = (N(\delta))^{-1/2} \text{ if } K \text{ is } p\text{-adic}. \end{cases}$$

different (↔ trace form!)

Theorem.

For $f \in L^1(K^+, \mathbb{C})$, conti., &

Define

$$\hat{f}(\chi) = \int_G f(\xi) \cdot e^{-2\pi i \langle \chi, \xi \rangle} d\xi.$$

If $f \in L^1(K^+)$, $\hat{f} \in L^1(K^+)$,

then

$$f(\xi) = \int \hat{f}(\eta) \cdot e^{2\pi i \Lambda(\eta\cdot\xi)} d\eta$$

$$= \hat{\hat{f}}(-\xi)$$

(pf) Pick a nice f
and take its Fourier transform.

$\boxed{\text{ゟ}}$

✓ multiplicative thy? ← no trace!

different !!

quasi-characters

Multiplicative Side

Def A <u>quasi-character</u> $c : K^+ \longrightarrow \mathbb{C}^\times$.
is just a conti. gp. hom.

α-case?

Def. c is <u>unramified</u> if it is trivial on $\mathcal{U}_K = \mathcal{O}_K^*$
$|\xi| = 1$

Any quasi-char. can be written
$$c(\alpha) = \tilde{c}(\tilde{\alpha})\,|\tilde{\alpha}|^s$$

where $\alpha = \hat{\alpha} \cdot \rho = \begin{cases} \rho > 0 & \text{if } K \text{ is real.} \\ \rho \text{ is a power of } \pi. & \text{when } K \text{ is } p\text{-adic.} \end{cases}$

unit part

$\tilde{c}$ is a character $\mathcal{U}_K \to \mathbb{C}^\times$.