9-29-2009 Ander Tate Thesis 10:00AM.

Recall quasi-characters.

$K = K_v$ cpltion of # field. $\rightsquigarrow$ multi. str. want to understand.
$= K_p$

Def. A quasi character c is just a conti. hom.

$$c : K^\times \longrightarrow \mathbb{C}^\times.$$

make sense if p-adic.

Def. c is unramified if c is trivial on $\mathcal{U}_K = \mathcal{O}_K^\times$.
$\qquad = \{u \in K : |u| = 1\}$.

lemma. The unram. quasichar. are

$$c(\alpha) = |\alpha|^s \quad (= \exp(s \cdot \log|\alpha|))$$

where $\underset{\mathbb{C}\,!!}{s}$ is determined, uniquely by c if K, real, cplx.
modulo $2\pi i / \log(Nm(p))$ if K, p-adic.

the concept of Riemann surfaces occurs.

Thm. For any $\alpha \in K$, write

$$\alpha = u \cdot \rho$$

where $u \in \mathcal{U}$ and ρ is > 0 when p archimedian.
ρ is an integral power of an uniformizer π.

Thm. The quasi-char. are given by

$$c(\alpha) = \underset{\substack{\uparrow \\ \text{char. on } U_K}}{\tilde{c}(u)} \cdot \underset{\substack{\uparrow \\ \text{unram!!}}}{|\alpha|^s} .$$

Def. $c \sim c'$ are equivalent q.-char.
if c/c' is unramified.
(live on the same sheet of the Riemann surface.)

$\rightsquigarrow$ Now we can control quasichar!!
$\downarrow$ move to
measures

Multiplicative Measures.

If $g \in L(K^\times)$,

$$g(\xi) \cdot |\xi|^{-1} \in L(K^+ \setminus \{0\})$$

$$\Phi(g) = \int_{K^+ \setminus \{0\}} g(\xi) \cdot |\xi|^{-1} \, d\xi .$$

If $h(\alpha) = g(\beta \cdot \alpha) \quad (\beta \in K^\times)$
$\Phi(h) = \Phi(g)$ } translation invariant.

There exists $d\alpha$ (a measure on $K^\times$)

s.t.

$$\Phi(g) = \int_{K^\times} g \, d_1\alpha.$$

$\mu(\beta \cdot M) = \underset{\text{mg}}{|\beta|} \cdot \mu(M)$!

We normalize $d_1\alpha$ so that

$$d\alpha = d_1\alpha = \frac{d\xi}{|\xi|} \quad \text{if } p \text{ archi.}$$

$$d\alpha = \underbrace{\frac{Np}{Np-1}}_{\text{geometric sum.}} d_1\alpha = \frac{Np}{Np-1} \cdot \frac{d\xi}{|\xi|} \quad \text{if } p \text{ finite.}$$

$$\int_{\mathcal{O}_K} d\xi = N\delta^{-1/2}$$

$$\int_{\underset{\sim}{U_K}} d\alpha = N \cdot \delta^{-1/2}. \qquad (\text{in adèles, almost } 1.)$$

U annuli

<u>Lemma.</u> $\displaystyle\int_{U_K} d\alpha = N \cdot \delta^{-1/2}.$

(pf) $\displaystyle\int_{U_K} d\alpha = \int_{U_K} |\xi|^{-1} d\xi = \int_{U_K} d\xi$

$$N\delta^{-1/2} = \int_{\mathcal{O}_K} d\xi = \sum_{n \geq 0} \int_{\underbrace{A_n}_{\text{annuli with val } n.}} d\xi = \underbrace{\sum_{n \geq 0}^{\infty} (N p)^{-n}}_{\text{geom. series}} \cdot \int_{U_K} d\xi.$$

$$= \frac{1}{1 - 1/Np} \cdot \int_U d\xi$$

(local ζ-ftns.
- analytic ftn of q-char ~~on~~ some domain)

<u>Def</u> A quasicharacter $c(\alpha) = c(u) \cdot |\alpha|^s$.
has exponent $\sigma = Re(s)$.

We will define a ζ-ftn. in the domain $\sigma > 0$.

Let $\mathfrak{Z}$ be the set of conti. ftns. on K^+.

satisfying

$\mathfrak{Z}1)$ f and $\hat{f}$ are both conti. and $\in L^1(K^+)$.

$\mathfrak{Z}2)$ $f(\alpha)\cdot|\alpha|^\sigma$ and $\hat{f}(\alpha)\cdot|\alpha|^\sigma \in L^1(K^\times)$ $\underline{\forall \sigma > 0}$

<u>Def.</u> Given $f \in \mathfrak{Z}$, c, a quasichar of exponent $\sigma > 0$,

$$\zeta(f, c) = \int_{K^\times} f(\alpha) \cdot c(\alpha) \cdot d\underset{\sim}{\alpha}.$$

multiplicative

(containing the complex variable)

convergence?
ensure
the integrability !!

$$\hat{c}(\alpha) = c^{-1}(\alpha) \cdot |\alpha|.$$

<u>Main Theorem.</u> $\zeta(f,c)$ has analytic conti. to all quasi-char.
given by
the ftn'l eqtn.

$$\zeta(f, c) = \rho(c) \cdot \zeta(\hat{f}, \hat{c}) \qquad (*)$$

indep. of f !!

where $\rho(c)$ is a meromorphic function of c
(fudge factor)

given by $(*)$

where $0 <$ exponent of $c < 1$.

and analytically continued to all quasi-char.

q.e.d.

<u>lemma</u>. For c of exponent $0 < \sigma < 1$ and $f, g \in \mathfrak{Z}$,

$$\zeta(f, c) \cdot \zeta(\hat{g}, \hat{c}) = \zeta(\hat{f}, \hat{c}) \cdot \zeta(g, c)$$

($\leadsto$ the ratio same !!)

Pick g cleverly.

In the case that K is real.

$\mathbb{R} \Rightarrow$ only 2 !!

$$c(\alpha) = \begin{cases} |\alpha|^s \\ \underset{\operatorname{sgn}(\alpha)}{\underset{\uparrow}{\alpha}}|\alpha|^s \end{cases}$$

The unram. char. is easy

Take. $g(\xi) = e^{-\pi \xi^2}$.

$\hat{g}(\xi) = e^{-\pi \xi^2}$.

$$\zeta(g, |\alpha|^s|) = \int_{K^\times} e^{-\pi\alpha^2} |\alpha|^s \, d\alpha.$$

$$= 2 \cdot \int_0^\infty e^{-\pi\alpha^2} \cdot \alpha^s \cdot d\alpha. \quad , \quad \alpha \rightsquigarrow t$$

$$= 2\int_0^\infty e^{-\pi t^2} \cdot t^s \cdot |t|^{-1} \cdot dt.$$

$$= 2 \cdot \int_0^\infty e^{-\pi t^2} \cdot t^{s-1} \, dt.$$

Let $r = \pi \cdot t^2$

$$= \pi^{-\frac{s}{2}} \cdot \int_0^\infty e^{-\pi t^2} \cdot (\pi \cdot t^2)^{s/2 - 1} \, \pi t \, dt$$

$$= \pi^{-s/2} \int_0^\infty e^{-r} \cdot r^{s/2 - 1} \, dr.$$

$$= \pi^{-s/2} \cdot \Gamma\left(\frac{s}{2}\right)$$

Since $\hat{g}(\xi) = g(\xi)$

$$\zeta(\hat{g}, |\alpha|^{1-s}) = \zeta(g, |\alpha|^{1-s})$$

$$= \pi^{-\frac{1-s}{2}} \cdot \Gamma\left(\frac{1-s}{2}\right)$$

$$\rho(c) = \frac{\zeta(g, c)}{\zeta(\hat{g}, \hat{c})} = 2^{1-s} \cdot \pi^{-s} \cdot \cos\left(\frac{\pi s}{2}\right) \cdot \Gamma(s)$$