MA 123 – Calculus I
Exam #1 Solutions

1. (15 points)
A particle is moving along a line. After \( t \) seconds, the distance the particle has travelled equals \( s(t) = t^3 + t \) meters.

(a) What is the average speed of the particle over the interval \( t = 1 \) to \( t = 2 \)?

Solution:

Average speed \( = \frac{s(2) - s(1)}{2 - 1} = \frac{2^3 + 2 - (1^3 + 1)}{1} = 8 \text{m/s}. \)

(b) Set up the limit that expresses the instantaneous speed of this particle at \( t = 1 \) second. (You do not need to evaluate this limit.)

Solution:

\[
\text{Instantaneous speed} = \lim_{h \to 0} \frac{s(1 + h) - s(1)}{h} = \lim_{h \to 0} \frac{(1 + h)^3 + (1 + h) - (1^3 + 1)}{h}
\]
2. (20 points)

(a) Let \( f(x) \) be the function defined by

\[
\begin{align*}
  f(x) &= \begin{cases} 
    \frac{2x}{x^2 + 2} & x < 1 \\
    x^2 + 1 & x \geq 1
  \end{cases},
\end{align*}
\]

Is \( f(x) \) continuous at \( x = 1 \)? Justify your answer.

Solution: At \( x = 1 \),

\[
\lim_{x \to 1^-} \frac{2x}{x^2 + 2} = \frac{2 \cdot 1}{1^2 + 2} = \frac{2}{3}
\]

and

\[
\lim_{x \to 1^+} \frac{x^2 + 1}{2x^2 + 1} = \frac{1^2 + 1}{2 \cdot 1^2 + 1} = \frac{2}{3}.
\]

Since the left-sided limit and the right-sided limit are the same and equal to \( f(1) \), the function is continuous at \( x = 1 \).

(b) What is the value of \( \lim_{x \to \infty} f(x) \)? Show your work.

Solution: Since we are considering \( x \to \infty \), we certainly have that \( x \geq 1 \) and so we only use the second case in the definition of the function.

\[
\lim_{x \to \infty} \frac{x^2 + 1}{2x^2 + 1} = \lim_{x \to \infty} \frac{x^2 + 1}{2x^2 + 1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}
\]

\[
= \lim_{x \to \infty} \frac{1 + \frac{1}{x^2}}{2 + \frac{1}{x^2}}
\]

\[
= \frac{1 + \lim_{x \to \infty} \frac{1}{x^2}}{2 + \lim_{x \to \infty} \frac{1}{x^2}}
\]

\[
= \frac{1 + 0}{2 + 0} = \frac{1}{2}
\]
3. **(15 points)**
Find the equation of the tangent line at \( x = 2 \) for the function \( y = \frac{1}{x+1} \).

(Show all work; it's not enough to use derivative formulas not discussed in class.)

**Solution:** If \( f(x) = \frac{1}{x+1} \), then

\[
\begin{align*}
    f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
    &= \lim_{h \to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \\
    &= \lim_{h \to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \\
    &= \lim_{h \to 0} \frac{\frac{3-(x+3+h)}{3(x+1)(x+h+1)}}{h} \\
    &= \lim_{h \to 0} \frac{-h}{3(x+3+h)h} \\
    &= \lim_{h \to 0} \frac{-1}{3(x+3+h)} \\
    &= \frac{-1}{9}.
\end{align*}
\]

So the equation of the tangent line at \( (2, \frac{1}{3}) \) is:

\[
y - \frac{1}{3} = \frac{-1}{9}(x - 2).
\]
4. **(20 points)**

On the axes below, sketch the graph of a function $f(x)$ with the following properties:

(a) $\lim_{x \to \infty} f(x) = 1$

(b) $\lim_{x \to -\infty} f(x) = -2$

(c) $\lim_{x \to -2^-} f(x) = 3$

(d) $f(x)$ has a jump discontinuity at $x = -1$

(e) $f(x)$ has an infinite discontinuity at $x = 0$

(f) $f(x)$ has a removable discontinuity at $x = 1$

(g) $f(x)$ is not differentiable at $x = 2$
5. **(15 points)** Match the graph of each function on the left with the graph of its derivative on the right. (Drawing an arrow from one graph to another will suffice. No justification is needed.)

(a) This graph has exactly two horizontal tangents which means the derivative has exactly two roots. This property is only satisfied by the third graph.

(b) This graph is increasing and thus the derivative is positive everywhere. This is only satisfied by the first graph.

(c) This function is not differentiable at \( x = 0 \). The only graphs of the derivatives with a point missing in their domain are the second and fourth. The original function is increasing and then decreasing and so its derivative is positive and then negative. This is only satisfied by the second graph.
6. (30 points)
For each statement, circle either TRUE or FALSE and write a sentence or two justifying your answer.

(a) If \( f(x) \) is continuous at \( x = 0 \), then \( f(x) \) is differentiable at \( x = 0 \).

FALSE: The graph of \( f(x) \) might have a corner at \( x = 0 \) and be continuous but not differentiable. For example, \( f(x) = |x| \).

(b) If \( \lim_{x \to 0} f(x) = \infty \) and \( \lim_{x \to 0} g(x) = -\infty \), then \( \lim_{x \to 0} f(x) + g(x) = 0 \).

FALSE: Infinity is not a number and the limit laws are not valid for infinite limits. For example, if \( f(x) = 1/x \) and \( g(x) = -1/x^2 \), then \( \lim_{x \to 0} 1/x = \infty \) and \( \lim_{x \to 0} -1/x^2 = -\infty \) but \( \lim_{x \to 0} 1/x + -1/x^2 = \lim_{x \to 0} (x - 1)/x^2 = -\infty \).

(c) Let \( f(x) \) be a function such that \( f'(x) > 0 \) for all \( x \). If \( f(2) = 3 \), then \( f(3) > 3 \).

TRUE: Since \( f'(x) > 0 \), \( f(x) \) is an increasing function. Thus, \( f(3) > f(2) = 3 \).

(d) A function could have infinitely many horizontal asymptotes.

FALSE: A function can have at most 2 horizontal asymptotes (whose value can be found by taking either the limit to \( \infty \) or to \( -\infty \)).

(e) The equation \( 2x^3 - 10x + 7 = 0 \) has a solution between 0 and 2.

TRUE: If \( f(x) = x^3 - 10x + 7 \), then \( f(0) = 7 \) and \( f(1) = -3 \). Thus, by the Intermediate Value theorem, \( f(x) \) has a solution between 0 and 1 (and thus between 0 and 2).