1. (a) **(15 points)** Write down an indefinite integral that can be solved by u-substitution, and evaluate this integral.

$$\int \sin(x^2)\cdot 2x\,dx = \int \sin(u)\,du$$

u-subst:

$$\left.\begin{aligned} u &= x^2 \\ du &= 2x\,dx \end{aligned}\right\}$$

$$= -\cos(u) + C$$

$$= -\cos(x^2) + C$$

(b) **(15 points)** Write down an improper integral that is convergent, and determine the value of this integral.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{x^2}\,dx$$

$$= \lim_{N\to\infty} -\frac{1}{x}\Big|_1^N$$

$$= \lim_{N\to\infty} -\cancel{\frac{1}{N}}^{\,0} + 1 = 1$$

2. **(30 points)** Evaluate **ONE** of the following three integrals:

- $\int \sin^2(x)\cos^3(x)\,dx$
- $\int x^2 e^x\,dx$
- $\int \sin(x) e^x\,dx$

$$\int \sin^2(x)\cos^3(x)\,dx = \int \sin^2(x)\left(1-\sin^2(x)\right)\cos(x)\,dx$$

u-Subst
$u = \sin(x)$
$du = \cos(x)\,dx$

$$= \int u^2(1-u^2)\,du$$

$$= \int u^2 - u^4\,du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C$$

- $$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$

IBP: $\boxed{\begin{array}{l} u = x^2 \Rightarrow du = 2x\,dx \\ dv = e^x dx \Rightarrow v = e^x \end{array}}$

$$= x^2 e^x - 2\underbrace{\int x e^x \, dx}_{}.$$

Do this one by IBP

$$\int x e^x \, dx = x e^x - \int e^x dx = x e^x - e^x + c$$

IBP $\boxed{\begin{array}{l} u = x \Rightarrow du = dx \\ dv = e^x \Rightarrow v = e^x \end{array}}$

Substituting back into the original equation gives:

$$\int x^2 e^x \, dx = x^2 e^x - 2\left(x e^x - e^x + c\right)$$

$$\boxed{= x^2 e^x - 2x e^x + 2e^x + c'}$$

$$\underbrace{\int \sin(x)\, e^x \, dx}_{I} = e^x \sin(x) - \underbrace{\int \cos(x)\, e^x \, dx}_{\text{Do this with IBP}}$$

IBP: $\boxed{\begin{array}{l} u = \sin(x) \Rightarrow du = \cos(x)\, dx \\ dv = e^x\, dx \Rightarrow v = e^x \end{array}}$

$$\int \cos(x)\, e^x \, dx = e^x \cos(x) + \int \sin(x)\, e^x \, dx$$

$$\int \sin(x)\, e^x \, dx = e^x \cos(x) + I.$$

IBP $\boxed{\begin{array}{l} u = \cos(x) \Rightarrow du = -\sin(x)\, dx \\ dv = e^x dx \Rightarrow v = e^x \end{array}}$

Substituting back in gives.

$$I = e^x \sin(x) - \left(e^x \cos(x) + I \right)$$

$$\Rightarrow 2I = e^x \sin(x) - e^x \cos(x)$$

$$\Rightarrow \boxed{I = \frac{e^x (\sin(x) - \cos(x))}{2}}$$

3. (a) (20 points) We know that

$$\int_1^3 \frac{1}{x}\,dx = \ln(3).$$

Show that ln(3) is smaller than 1.5 by approximating the above integral (using any methods you wish). Be sure to fully justify your answer.

Method A:

For example, by the midpoint rule with $n=2$, we get

$$M_2 = 1\cdot\left(\frac{1}{\frac{3}{2}} + \frac{1}{\frac{5}{2}}\right) = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}.$$

Clearly, $\frac{16}{15} < 1.5$, but to see that $\ln(3) < 1.5$, we need to check that our ~~error~~ ~~estimate~~ estimate is good enough. Using error bounds gives

$$|E_{M_2}| \leq \frac{K(b-a)^3}{24n^2} = \frac{K\cdot 2^3}{24\cdot 2^2} = \frac{K}{12}$$

where $K \geq |f''(x)|$ with x in $[1,3]$.

To find K, we have $f'(x) = \frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$.

On $[1,3]$, $f''(x) = \frac{2}{x^3}$ is decreasing, so its maximum

Cont.

is achieved at $x=1$. We can then take

$K = f''(1) = 2$.

Thus, $|E_{M_2}| \leq \frac{2}{12} = \frac{1}{6}$.

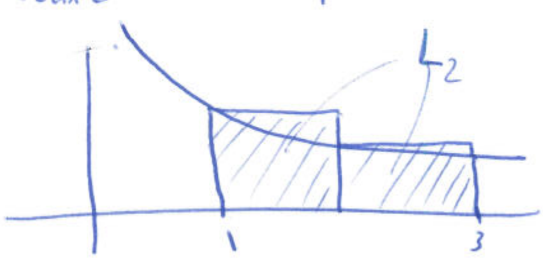
Since M_2 is within $\frac{1}{6}$ of $\ln(3) = \int_1^3 \frac{1}{x}\,dx$,

we have $\ln(3) < M_2 + \frac{1}{6} = \frac{16}{15} + \frac{1}{6} = \frac{37}{30} < 1.5$.

Thus, $\ln(3) < 1.5$ as desired.

Method B: We give an upper bound of $\int_1^3 \frac{1}{x}\,dx = \ln(3)$.

Namely, the left endpoint rule with $n=2$ gives an upper bound as explained in the picture (since $\frac{1}{x}$ is a decreasing function).



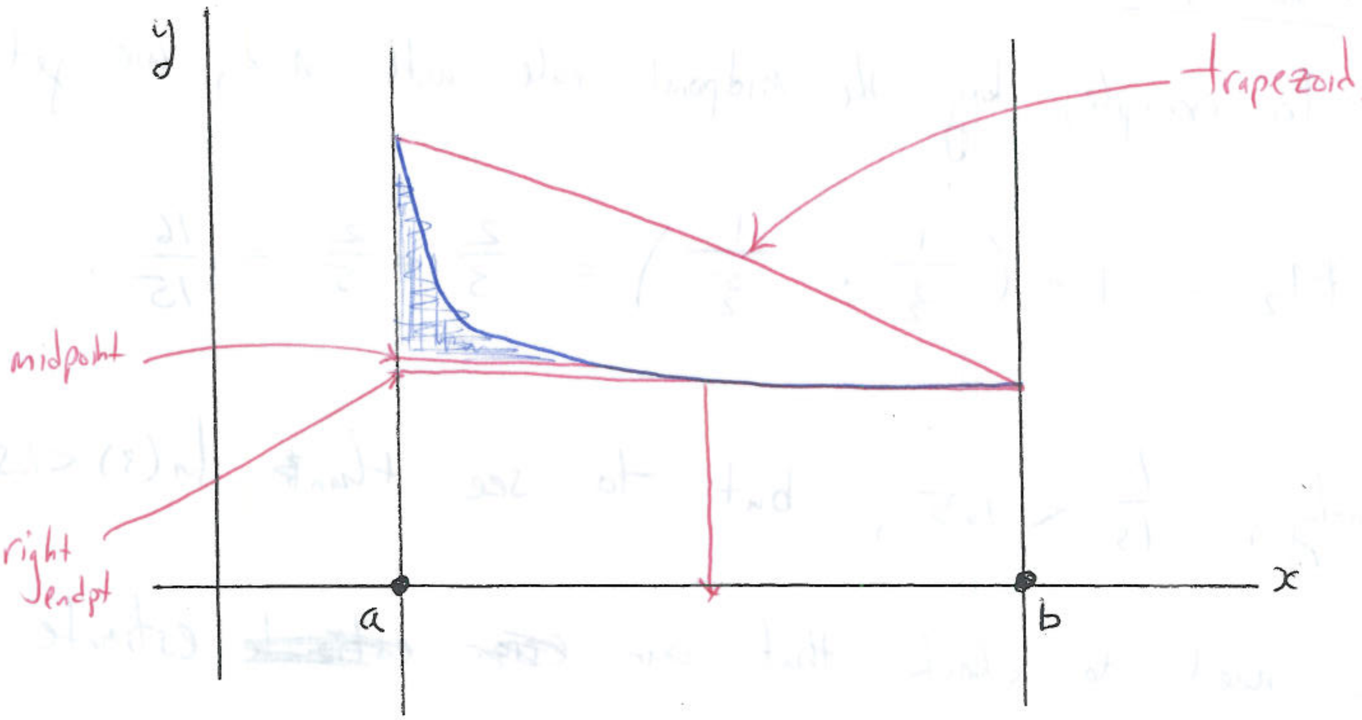
So $L_2 > \int_1^3 \frac{1}{x}\,dx = \ln 3$. Since $L_2 = 1 + \frac{1}{2} = \frac{3}{2}$,

we get $\ln(3) < \frac{3}{2}$.

(b) **(10 points)**
On the axes below, sketch the graph of a function $y = f(x)$ on the interval $[a, b]$ for which **ALL** of the following are true:

- the trapezoid rule with $n = 1$ gives an overestimate for $\int_a^b f(x)\,dx$;
- the midpoint rule with $n = 1$ gives an underestimate for $\int_a^b f(x)\,dx$;
- the right endpoint rule with $n = 1$ gives an underestimate for $\int_a^b f(x)\,dx$;

Justify your answers in the space provided below the graph.



- the trapezoid estimate is an overestimate as it ~~el~~ sits entirely above the graph. (the function is concave-up)
- the midpt estimate is an underestimate as its rectangle misses the larger ~~reg~~ shaded region on the left.
- the right endpt estimate is an underestimate since the function is descreasing and its rectangle is thus below the graph.

4. **(30 points)** Evalute the following integral:

$$\int_1^e \frac{\ln(\ln x)}{x}\,dx.$$

[Note that this is an improper integral as ln(ln(1)) is undefined.]

$$\int_1^e \frac{\ln(\ln x)}{x}\,dx = \int_{u=0}^{u=1} \ln(u)\,du$$

$$\boxed{\begin{aligned} &\underline{u\text{-Subst}}: \quad u = \ln x \\ &\qquad\qquad\quad du = \tfrac{1}{x}\,dx \end{aligned}}$$

Let's first do the ~~improper~~ indefinite integral:

$$\int \ln(u)\,du = u\ln u - \int u\left(\tfrac{1}{u}\right)du = u\ln u - \int du$$

$$\boxed{\begin{aligned} \underline{IBP}: \quad & a = \ln u \Rightarrow da = \tfrac{1}{u}\,du \\ & db = du \Rightarrow b = u \end{aligned}}$$

$$= u\ln u - u + C$$

Returning to the original (improper) integral:

$$\int_0^1 \ln(u)\,du = \lim_{t\to 0^+}\int_t^1 \ln(u)\,du = \lim_{t\to 0^+}(u\ln u - u)\Big|_t^1$$

$$= \lim_{t\to 0^+}(1\cdot\ln 1 - 1) - (t\ln t - t) = \lim_{t\to 0^+} -1 - t\ln t + t$$

over

$$= -1 \cancel{-} \lim_{t \to 0^+} t \ln t + \lim_{t \to 0^+} t$$

$$= -1 - \lim_{t \to 0^+} \frac{\ln t}{\frac{1}{t}}$$

$$= -1 - \lim_{t \to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = -1 + \lim_{t \to 0^+} t$$

L'Hospital rule

$$= -1 + 0$$

$$= \boxed{-1}$$