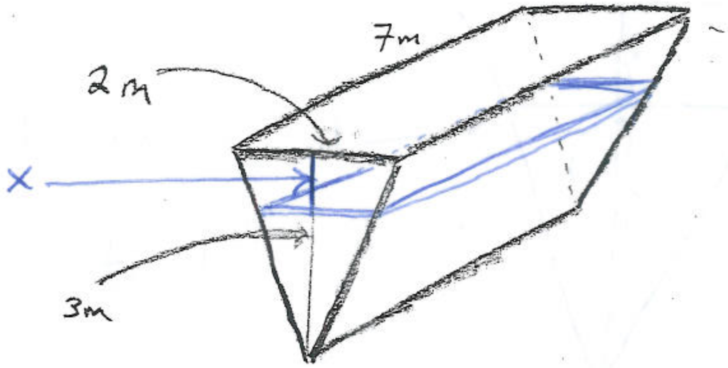
1. **(30 points)** The tank shown below is filled with water. Write down a definite integral that describes how much work it takes to pump all of the water out of the top of this tank.

- You may use the approximation $g = 10$ m/s$^2$. The density of water is 1000 kg/m$^3$.
- You do not need to evaluate this integral.




We consider a horizontal slice at depth $x$ and with thickness $\Delta x$. If $A(x)$ is the area of ~~this~~ the top of this slice, then its volume is $A(x)\,\Delta x$.

Thus, its mass is $A(x) \cdot 1000 \cdot \Delta x$,

and its weight is $A(x) \cdot 1000 \cdot g \cdot \Delta x \approx A(x) \cdot 10^3 \cdot (9.8)\,\Delta x$

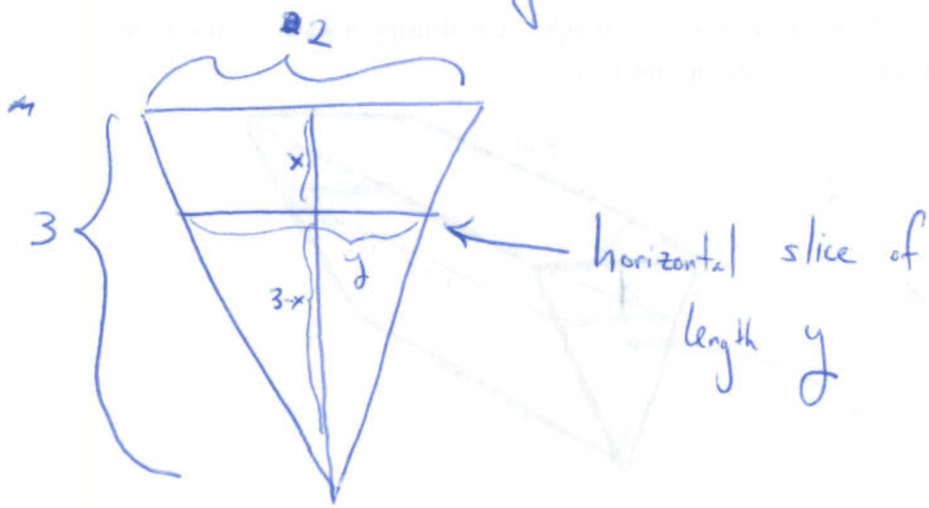
The total work to pump the water out of the tank is then

$$\int_0^3 A(x) \cdot 10^3 \cdot g \cdot x \, dx .$$

(Here $A(x)\,10^3 g$ represents the force and $x$ represents the distance.)

We must determine $A(x)$. over

We draw the front of the tank




By similar $\Delta$'s, $\dfrac{y}{2} = \dfrac{3-x}{3}$

$$\Rightarrow \quad y = \frac{2}{3}(3-x)$$

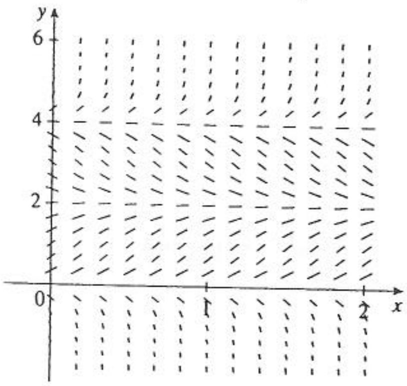
Since $A(x) = y \cdot 7 = \frac{2}{3}(3-x)\cdot 7$,

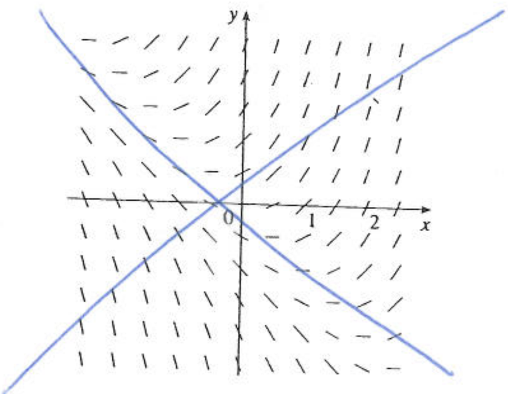
the final answer is:

$$\boxed{\int_0^3 \frac{14}{3}\cdot 10^3 \cdot g\,(3-x)\,x\,dx}$$

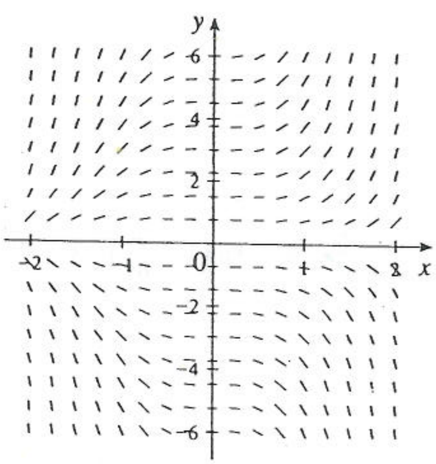
2. (a) **(15 points)** Draw an arrow from each direction field to the differential equation that defines it.

A

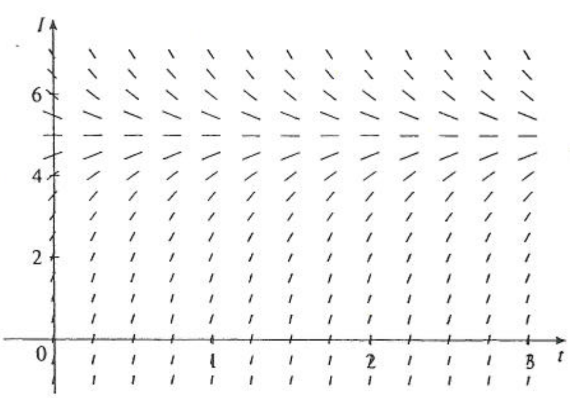



B




C




$y' = x + y - 1$ 1

$y' = y^2$ 2

$y' = (y-2)(y-4)$ 3

$y' = 15 - 3y$ 4

$y' = y(y-2)(y-4)$ 5

$y' = x^2y$ 6

$y' = (y-5)e^y$ 7

(b) **(15 points)** Draw an arrow from each data table to the differential equation that best models this data.

A

| $t$ | $y(t)$ |
|---|---|
| 0 | 900 |
| 1 | 300 |
| 2 | 100 |
| 3 | 33.3 |
| 4 | 11.1 |
| 5 | 1.23 |
| 6 | 0.41 |
| 7 | 0.14 |
| 8 | 0.05 |
| 9 | 0.02 |
| 10 | 0.01 |

B

| $t$ | $y(t)$ |
|---|---|
| 0 | 2 |
| 1 | 4 |
| 2 | 9 |
| 3 | 17 |
| 4 | 28 |
| 5 | 40 |
| 6 | 51 |
| 7 | 57 |
| 8 | 61 |
| 9 | 62 |
| 10 | 63 |

C

| $t$ | $y(t)$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |
| 6 | 64 |
| 7 | 128 |
| 8 | 256 |
| 9 | 512 |
| 10 | 1024 |

$y' = k(y - 64)\,,\ k > 0$ 1

$y' = ky,\ k > 0$ 2

$y' = ky^2,\ k < 0$ 3

$y' = k2^y,\ k > 0$ 4

$y' = ky\left(1 - \frac{y}{64}\right),\ k > 0$ 5

$y' = ky,\ k < 0$ 6

# Explanations to 2(a)

A: $y'$ vanishes when $y = 0$, $2$ and $4$.

The only possible answer is then 5 $y' = y(y-2)(y-4)$.

[The $2^{nd}$ direction field was thrown out. It should have matched with $y' = x + y$ (which is not listed).]

B $y'$ vanishes when $y = 0$. The only possible answers are then 2 and 6. Since $y' < 0$ for $y < 0$, the only possible answer is 6.

C: $y'$ vanishes when $y = 5$. The only possible answers are then 4 and 7. Since $y' < 0$ for $y < 5$, the only possible answer is 4.

Explanations to 2(b)

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A: $y(t)$ is decaying exponentially (by a factor of 3 each time) and so $y' = ky$ with $k<0$.

6

[In fact, $k = -3$]

---

B: $y(t)$ is increasing quickly and then leveling off towards 64. This growth is modeled by

$$y' = ky\left(1 - \frac{y}{64}\right),$$

"population ~~with~~ growth with a fixed capacity"

---

C $y(t)$ is increasing exponentially (by a factor of 2 each time). Thus $y' = ky$ with $k>0$.

[In fact, $k=2$]

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3. **(30 points)** Initially a water tower contains 1000 liters of pure water. Two valves are then opened, one allowing a solution of water and fluoride with a concentration of 7 grams of fluoride per liter of water to flow into the system at a rate of 10 liters per minute, and the other valve allows the solution in the tank to be drained at 10 liters per minute. We assume that the solution is mixed constantly so that the fluid in the tank is homogeneous.

(a) Determine how much fluoride is in the tank after $t$ minutes.

Let $F(t)$ denote the amount of fluoride in the solution at time $t$. Then

$$F'(t) = \text{rate in} - \text{rate out}$$

$$\Rightarrow F'(t) = (7g/L)\cdot(10\, L/min) - (F(t)/1000)\cdot(10\, L/min)$$

$$\Rightarrow F'(t) = 70 - \frac{F(t)}{100} = \frac{7000 - F(t)}{100}$$

$$\Rightarrow \frac{dF}{dt} = \frac{7000 - F(t)}{100}$$

$$\Rightarrow \int \frac{dF}{7000 - F(t)} = \int \frac{dt}{100}$$

$$\Rightarrow -\ln(7000 - F(t)) = \frac{t}{100} + C$$

$$\Rightarrow 7000 - F(t) = e^{-(\frac{t}{100} + C)} = Ae^{\frac{-t}{100}}$$

over

$$\Longrightarrow \quad F(t) = 7000 - A e^{-\frac{t}{100}}$$

To determine $A$, take $t=0$. Since $F(0)=0$ (the water is <u>pure</u>)

we have $$0 = 7000 - A \cdot e^{0}$$

$$\Longrightarrow \quad A = 7000$$

$$\Longrightarrow \quad F(t) = 7000 - 7000\, e^{-\frac{t}{100}}$$

$$\Longrightarrow \quad \boxed{F(t) = 7000\left(1 - e^{-t/100}\right)}$$

(b) Use your answer from the previous part to determine what happens to the amount of fluoride in the tank as $t$ tends to infinity.

$$\lim_{t \to \infty} F(t) = \lim_{t \to \infty} 7000\left(1 - e^{\frac{-t}{100}}\right) = \underline{7000} \text{ g}$$

(c) Explain in a sentence or two why your answer makes logical sense compared to what one would expect to happen to this water solution in the long run.

In the long run, the tank will be nearly entirely full of the new water running into the tank. ~~If~~ Since this water contains 7 g/L, if we had 1000 L of such water, we would indeed expect 7000 g of fluoride!

4. **(30 points)** For each question ciricle either "true", "false" or "I don't know". A correct answer scores you 7 points. An incorrect answer scores you 0 points. An "I don't know" answer scores you 3 points.

Note that no justification is required.

(a) The functions $y = e^{\sin x} + C$ are solutions of the differential equation $y' = y \cos x$.

TRUE                I DON'T KNOW                FALSE

(b) A 100-foot rope is hanging off the top of a building. The amount of work it takes to pull 50 feet of the rope to the top of building is half the amount of work required to pull the entire rope to the top of the building.

TRUE                I DON'T KNOW                FALSE

(c) Consider the solid obtained by rotating the region bounded by $y = x + \sin(x)$, $y = 0$, and $x = \frac{\pi}{2}$ around the $y$-axis. The volume of this region is given by

$$2\pi \int_0^{\frac{\pi}{2}} x(x + \sin x)\, dx.$$

TRUE                I DON'T KNOW                FALSE

(d) Consider the the differential equation

$$y' = y(y-2).$$

If $y$ is a solution of this equation with $y(0) = 1$, then $\lim_{t \to \infty} y(t) = 2$.

TRUE                I DON'T KNOW                FALSE

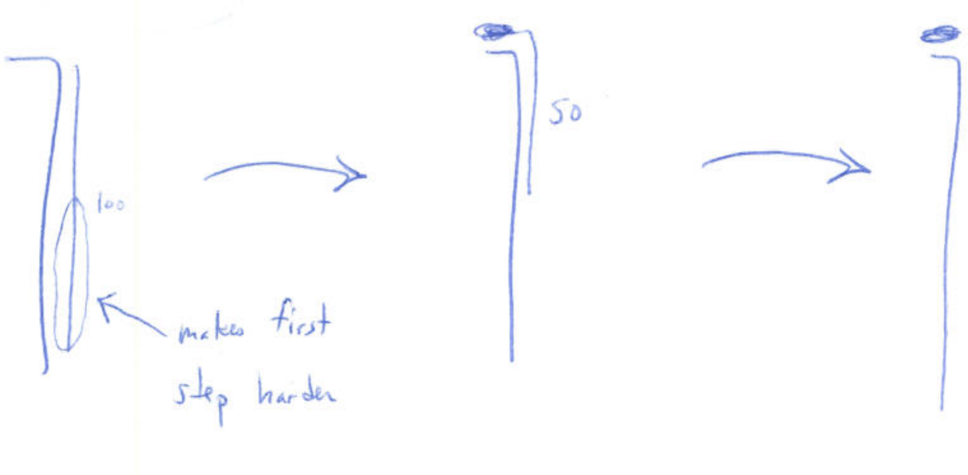
# Explanations to T/F

(F)

(1) $y = Ce^{\sin(x)}$ are the solutions to this diff eqn, not $e^{\sin(x)} + C$. Note that if $y = e^{\sin(x)} + C$, then $y' = e^{\sin(x)} \cdot \cos(x) = (y - C)\cos(x)$ and not $y \cdot \cos(x)$.

(F)

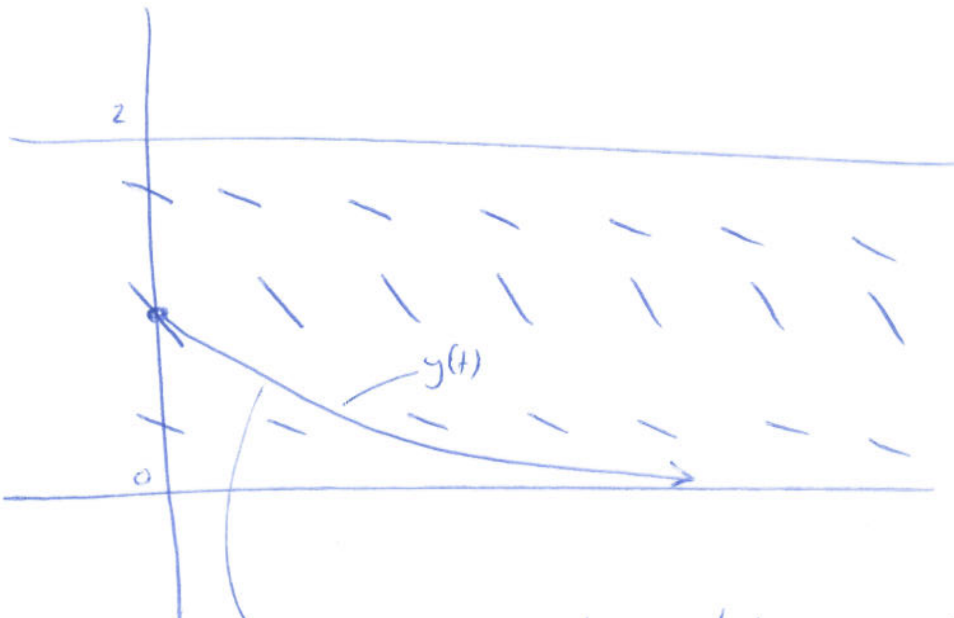
(2) Pulling up 100 feet of rope can be thought of taking 2 steps: first pull up 50 ft of rope and then another 50 feet of rope. But the first step is clearly harder than the second since one is also lifting the bottom half of the rope.



(3) True Using cylindrical shells we obtain

$$\int_0^{\frac{\pi}{2}} 2\pi r h \, dx = \int_0^{\frac{\pi}{2}} 2\pi x (x + \sin x) \, dx$$

(4) False $y' = y(y-2)$ has as a direction field




This is the solution with $y(0) = 1$.

But $\lim_{t \to} y(t) = 0$.