MA124 Midterm 1 Solutions

1. $\int \frac{x^2}{\sqrt{1-x^3}}dx.$

   To solve this, take $t = 1 - x^3$, then $-\frac{dt}{3} = x^2 dx$. The integral becomes

$$\int \frac{-dt}{3\sqrt{t}} = \frac{-2\sqrt{t}}{3} + C = \frac{-2\sqrt{1-x^3}}{3} + C.$$

2. $\int_0^{\pi/2} \sin^2(x)\cos^3(x)dx$

   First solve the indefinite integral using the substitution $t = \sin(x)$ with $dt = \cos(x)dx$.

$$\begin{aligned}\int \sin^2(x)\cos^3(x)dx &= \int \sin^2(x)\cos^2(x)\cos(x)dx = \int \sin^2(x)(1-\sin^2(x))\cos(x)dx \\ &= \int t^2(1-t^2)dt = \frac{t^3}{3} - \frac{t^5}{5} = \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5}\end{aligned}$$

   Thus

$$\begin{aligned}\int_0^{\pi/2} \sin^2(x)\cos^3(x)dx &= \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5}\Big|_0^{\pi/2} \\ &= \left(\frac{\sin^3(\pi/2)}{3} - \frac{\sin^5(\pi/2)}{5}\right) - \left(\frac{\sin^3(0)}{3} - \frac{\sin^5(0)}{5}\right) \\ &= \frac{1}{3} - \frac{1}{5}\end{aligned}$$

   as $\sin(\frac{\pi}{2}) = 1$ and $\sin(0) = 0$.

3. $\int_0^1 x^2 e^{-x} dx.$

   First solve the indefinite integral using integration by parts with $u = x^2$ and $dv = e^{-x}dx$.

$$\int x^2 e^{-x} dx = -x^2 e^{-x} + 2\int x e^{-x} dx.$$

   Apply integration by parts to the last integral with $u = x$ and $dv = e^{-x}dx$.

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x}.$$

   Evaluating at 0 and 1 we get

$$\int_0^1 x^2 e^{-x} dx = (-e^{-1} - 2e^{-1} - 2e^{-1}) - (0 + 0 - 2) = -5e^{-1} + 2 = 2 - \frac{5}{e}.$$

4. Any decreasing function which is concave up will do. Indeed, the trapezoid rule always gives an overestimate for a concave up function. Similarly, the midpoint rule always gives an underestimate for a concave up function. (Draw some pictures to convince yourself of this!) Lastly, the right endpoint rule always gives an underestimate for a decreasing function.

5. Given that $\int_1^3 \frac{dx}{x} = \ln(3)$, show that $\ln(3) < 1.4$.

   There are many possible solutions to this question. Here is one: since $f(x) = \frac{1}{x}$ is concave up, the trapezoid rule will always give an overestimate. So, applying the trapezoid rule with $n = 1$ gives:

$$\ln(3) = \int_1^3 \frac{dx}{x} < \frac{\Delta x}{2}\cdot(f(1) + f(3)) = \frac{2}{2}\cdot\left(1 + \frac{1}{3}\right) = 4/3 = 1.\overline{3} < 1.4$$

   as desired.

6. Determine if the following improper integrals are convergent or divergent:

The first integral can be computed directly

$$\int_1^\infty \frac{dx}{\sqrt[3]{x}} = \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt[3]{x}} = \lim_{t\to\infty} \frac{3x^{\frac{2}{3}}}{2}\Big|_1^t = \infty.$$

Therefore, it is divergent.

The second integral is impossible to compute directly, so we must use the comparison principle.

For $x \geq 1$, we have $0 < e^{-x} < 1$ and so $0 \leq \frac{e^{-x}}{x^5} \leq \frac{1}{x^5}$. This implies that

$$\int_1^\infty \frac{e^{-x}}{x^5}dx \leq \int_1^\infty \frac{1}{x^5}dx.$$

Also

$$\int_1^\infty \frac{1}{x^5}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^5}dx = \lim_{t\to\infty} -\frac{x^{-4}}{4}\Big|_1^t = \frac{1}{4}.$$

Therefore, $\int_1^\infty \frac{1}{x^5}dx$ is convergent, and thus by comparison, the original integral $\int_1^\infty \frac{e^{-x}}{x^5}dx$ is convergent.

7. $\int_1^e \frac{\ln(\ln x)}{x}dx$

Use the substitution $s = \ln x$ with $ds = \frac{dx}{x}$. The limits change from 1 and e to 0 and 1.

$$\int_1^e \frac{\ln(\ln x)}{x}dx = \int_0^1 \ln s \; ds.$$

Apply integration by parts with $u = \ln s$ and $dv = ds$

$$\int \ln s \; ds = s\ln s - s.$$

Therefore

$$\begin{aligned}\int_0^1 \ln s\, ds &= \lim_{t\to 0} s\ln s - s\Big|_t^1 = \lim_{t\to 0}(1\ln 1 - 1) - (t\ln t - t) = -1 - \lim_{t\to 0} t\ln t + \lim_{t\to 0} t \\ &= -1 - \lim_{t\to 0} t\ln t = -1 - \lim_{t\to 0}\frac{\ln t}{\frac{1}{t}} \overset{LH}{=} -1 - \lim_{t\to 0}\frac{\frac{1}{t}}{\frac{-1}{t^2}} = -1 - \lim_{t\to 0} -t = -1 + 0 = -1.\end{aligned}$$