Solutions to Second Midterm

1. *Compute the volume of the solid obtained by rotating the region bounded by the graphs of $y = 1$, $y = e^x$ and $x = 1$ around the y-axis.*

   Using cylindrical shells, the height is $e^x - 1$ and the radius is x. Therefore
   $$V = \int_0^1 2\pi x(e^x - 1)\, dx.$$
   To solve this integral, we apply integration by parts with $u = x$ and $dv = (e^x - 1)dx$.
   $$\begin{aligned} \int_0^1 2\pi x(e^x - 1)\, dx &= 2\pi x(e^x - x)\Big|_0^1 - \int_0^1 2\pi(e^x - x)\, dx \\ &= 2\pi((e - 1 - e^x\Big|_0^1 + \frac{x^2}{2}\Big|_0^1)) \\ &= 2\pi(e - 1 - (e - 1) + \frac{1}{2}) = \pi. \end{aligned}$$

2. *Find the work required to pump all the water out of the top of the tank, given in the figure.*

   Let the variable y represent the height from the bottom of the tank. Take a horizontal slice at height y with thickness Δy. Using similarity of triangles we get, denoting by α the smaller side of the slice, that
   $$\frac{y}{\alpha} = \frac{3}{2},$$
   so $\alpha = \frac{2y}{3}$. The volume of the slice is
   $$\frac{2y}{3} 7\Delta y = \frac{14y}{3}\Delta y \quad m^3.$$
   The mass of the slice of water is
   $$1000 \cdot \frac{14y}{3} = \frac{14000y}{3}\Delta y \quad kg.$$
   To compute the force to be used on the slice, we multiply by gravity
   $$\frac{140000y}{3}\Delta y \quad N.$$

This slice has to be lifted $(3-y)$ m^3 so the work performed on the slice is

$$(3-y)\frac{140000y}{3}\Delta y \quad J.$$

Thus, the total work is given by

$$W = \int_0^3 (3-y)\frac{140000y}{3}dy.$$

3. *Match the vector fields with the appropriate differential equations*

   The second graph corresponds to $y' = x+y$. Note that $y' = 0$ on points where $x+y=0$. This implies that the vectors should be horizontal along the line $y = -x$.

   The third graph corresponds to $y' = x^2y$. Notice that $y' = 0$ when $x = 0$ or $y = 0$. This implies that the vectors should be horizontal along $x = 0$ and $y = 0$.

   The fourth graph corresponds to $y' = 15-3y$. Notice that $y' = 0$ when $15 - 3y = 0$. This implies that the vectors should be horizontal along the line $y = 5$.

4. *A tank with* 1000 *liters of pure water, starts receiving, at* 10 *liters per minute, a solution of water and fluoride with a concentration of 7 grams of fluoride per liter. At the same rate, the homogeneous fluid is drained from the tank. Compute the concentration of fluoride as a function of time and investigate its long term behavior.*

   Denote by $x(t)$ the concentration (in kilograms) of fluoride in the tank at time t (minutes). We have $x(0) = 0$. The amount of fluoride entering the tank is equal to $0.007 \cdot 10 = 0.07$ kilograms per minute. The amount of fluoride exiting the tank is $\frac{x(t)}{1000}10 = \frac{x(t)}{100}$ kilograms per minute.

   The initial value problem corresponding to this system is

$$\begin{aligned}\frac{dx}{dt} &= 0.07 - \frac{x(t)}{100}\\ x(0) &= 0.\end{aligned}$$

The equation can be rewritten using separation of variables, getting

$$\begin{aligned}
\frac{dx}{dt} &= \frac{7}{100} - \frac{x}{100} \\
\frac{dx}{dt} &= \frac{7-x}{100} \\
\frac{dx}{7-x} &= \frac{dt}{100}
\end{aligned}$$

Integrating, we get

$$-\ln|7-x| = \frac{t}{100} + C,$$

where C is a constant. Solving for x,

$$x(t) = 7 - ke^{-\frac{t}{100}}$$

where k is another constant. Since $x(0) = 0$, the final answer is

$$x(t) = 7 - 7e^{-\frac{t}{100}}.$$

(b) To determine what happens as t tends to infinity, we compute

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} 7 - 7e^{-\frac{t}{100}} = 7.$$

Therefore, the amount of fluoride in the tank tends to 7 kg.

(c) Note that this answer makes sense. Indeed, as t tends to infinity, the concentration in the tank should get closer and closer to the concentration of the solution added to this tank. Since this concentration is 0.007 kg/L and there are 1000 liters of solution, we should expect the amount of fluoride to tend to $0.007 \cdot 1000$ kg which equals 7 kg (matching the answer to part (b)).

5. *Decide wether the following statements are true or false*

(a) The average value of $f(x) = x^2$ on $[0, 1]$ is $\frac{1}{3}$. **TRUE**.

$$\mathrm{Av} = \frac{1}{1-0}\int_0^1 \mathrm{x}^2\,\mathrm{dx} = \frac{\mathrm{x}^3}{3}\Big|_0^1 = \frac{1}{3}.$$

(b) A 100-foot rope is hanging off the top of a building. The amount of work required to pull 50 feet of the rope to the top of the building is half the amount of work required to pull the entire rope to the top of the building. **FALSE**

The work required to lift the first half of the rope is larger than the second half of the rope. The reason for this is that when you are pulling the first half of the rope up, you also are lifting the second half of the rope up 50 ft. When pulling up the second half of the rope, this additional work is not done.

(c) Consider the graph obtained by rotating the region bounded by $y = x + \sin(x)$, $y = 0$ and $x = \frac{\pi}{2}$, around the y-axis. The volume of the region is given by $\pi \int_0^{\frac{\pi}{2}} (x + \sin(x))^2 \, dx$. **FALSE**

The only way to compute this volume is to use cylindrical shells. The radius is equal to x and the height is $x + \sin(x)$, so the volume is

$$2\pi \int_0^{\frac{\pi}{2}} x(x + \sin(x)) \, dx.$$

(d) Consider the differential equation $y' = y(y - 2)$. If y is a solution with $y(0) = 1$, then $\lim_{t\to\infty} y(t) = 2$. **FALSE**

We can see this by sketching the direction field. This differential equation has two equilibrium solutions given by $y = 0$ and $y = 2$. Moreover, between $y = 0$ and $y = 2$, we have that y' is negative. Thus all solutions in this range slope downwards and are attracted to the equilibrium solution $y = 0$. Thus, $\lim_{t\to\infty} y(t) = 0$.