Can you imagine a mathematician writing Moby Dick? Let my name be Ishmael, let the captain's name be Ahab, let the boat's name be Pequod, and let the whale's name be as in the title. – Barry Cipra

Honors Calculus – Math 129 – Fall 2009 – R. Pollack
HW #2

For a and b real numbers, we define the notation $a > b$ to mean $a + -b \in P$; that is, a is greater than b if $a + -b$ is a positive number.

1. Prove that
$$a > 0 \text{ if and only if } a \in P.$$
[Recall that "if and only if" statements are two statements in one. You need to prove both
   - if $a > 0$, then $a \in P$.
   - if $a \in P$, then $a > 0$.]

2. Let a, b and c be real numbers with $c > 0$ and $a > b$. Prove that
$$a \cdot c > b \cdot c.$$
[Hint: Write out carefully the definition of ">" in all of the givens and in what you want to prove.]

3. Let x be a positive real number which is smaller than 1. Prove that when you multiply x by itself the result is smaller than what you started with.
[Hint: First convert the above "English" sentence into mathematical notation. Determine precisely what is given and what needs to be proven.]

4. Prove that for all real numbers a and b,
$$a^2 - b^2 = (a + b) \cdot (a - b).$$

5. If a and b are positive real numbers, prove that $a > b$ implies $a^2 > b^2$.

6. If a and b are positive real numbers, prove that $a^2 > b^2$ implies $a > b$.

7. Find the least upper bounds and the greatest lower bounds (if they exist) of the following sets.
   (a) $\left\{\frac{1}{n} : n \in \mathbb{Z}, n > 0\right\}$;
   (b) $\left\{\frac{1}{n} : n \in \mathbb{Z}, n \neq 0\right\}$;
   (c) $\{x : x^2 + x - 1 \geq 0\}$;
   (d) $\{x : x^2 + x - 1 < 0\}$;
   (e) $\left\{\frac{1}{n} + (-1)^n : n \in \mathbb{Z}, n > 0\right\}$;

8. (Challenge question)
Let $S = \{x \in \mathbb{R} : x^2 < 2\}$. If α equals the least upper bound of S, prove that $\alpha^2 = 2$.
[Hint: (1) Why does S even have a least upper bound? (2) Rule out the possibilities that $\alpha^2 > 2$ and $\alpha^2 < 2$.]