Each problem that I solved became a rule which served afterwards to solve other problems.
—Descartes

Honors Calculus – Math 129 – Fall 2009 – R. Pollack
HW #4

Axiomatic proofs: The following proofs should be done with axiomatic rigor.
Recall that for $x \in \mathbb{R}$, we say x is positive if $x \in P$. Further, for $x \in \mathbb{R}$, we say that x is *negative* if $-x \in P$.

1. Prove the following:
   (a) If x is positive and y is negative, then $x \cdot y$ is negative.
   (b) If x and y are negative, then $x \cdot y$ is positive.
   (c) If $x \cdot y$ is positive and x is positive, then y is positive.

More questions:

2. Recall that for subsets A and B of $\mathbb{R}$, we define
$$A \cup B = \{x \in \mathbb{R} \mid x \in A \ \text{ or } \ x \in B\}$$
and
$$A \cap B = \{x \in \mathbb{R} \mid x \in A \ \text{ and } \ x \in B\}.$$
We further define A^c (read as A *complement*) by
$$A^c = \{x \in \mathbb{R} \mid x \text{ is not in } A\} = \{x \in \mathbb{R} \mid x \notin A\}.$$
Prove the following statements or give a counter-example:
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
   (c) $(A \cup B)^c = A^c \cap B^c$.
   (d) $(A \cap B)^c = A^c \cup B^c$.
   (e) $(A^c)^c = A$.
3. Let S be a non-empty subset of $\mathbb{R}$ which is bounded below. Prove that S has a greatest lower bound. [Hint: Let $-S$ denote the set $\{-x \bigm| x \in S\}$. Prove that $-S$ is bounded from above and that if x is the least upper bound of $-S$, then $-x$ is the greatest lower bound of S.]
4. Prove that the greatest lower bound of a set is unique.
5. Prove that
$$\inf \left\{ \frac{1}{x} \mid x \in \mathbb{R}, x > 0 \right\} = 0.$$
6. For any α in $\mathbb{R}$, prove that
$$\sup\{x \in \mathbb{R} \mid x < \alpha\} = \alpha.$$
7. Prove each of the following statements or give a counter-example to them. In each part, S and T are non-empty subsets of $\mathbb{R}$ which are bounded from above.
   (a) If $\sup S \leq \sup T$, then for every $s \in S$ and for every $t \in T$ we have $s \leq t$.
   (b) If for every $s \in S$ and for every $t \in T$ we have $s \leq t$, then $\sup S \leq \sup T$.