Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #10

Questions from Rudin:
Chapter 5: 1, 12

Additional questions:

1. Let X be a metric space and let p be a limit point of X. Prove the following basic properties of limits.
   (a) Let $f : X \to \mathbb{R}$ be the constant function defined by $f(x) = c$ for all $x \in X$. Prove $\lim_{x\to p} f(x) = c$.
   (b) Let $f, g : X \to \mathbb{R}$. If $\lim_{x\to p} f(x)$ exists and $\lim_{x\to p} g(x)$ exists, then $\lim_{x\to p} f(x) + g(x)$ exists and
   $$\lim_{x\to p} f(x) + g(x) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x).$$
   (c) Let $f, g : X \to \mathbb{R}$. If $\lim_{x\to p} f(x)$ exists and $\lim_{x\to p} g(x)$ exists, then $\lim_{x\to p} f(x) + g(x)$ exists and
   $$\lim_{x\to p} f(x) \cdot g(x) = \lim_{x\to p} f(x) \cdot \lim_{x\to p} g(x).$$

2. Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions and such that
   - $f(a) = g(a)$,
   - $f'(x) > g'(x)$ for all $x \in (a, b)$.

   Prove that $f(x) > g(x)$ for $x > a$.
   (Hint: Apply the mean value theorem to $h(x) = f(x) - g(x)$.)

3. Consider the exponential function $y = e^x$. Prove the following inequalities:
   (a) $x > 0 \implies e^x > 1 + x$
   (b) $x > 0 \implies e^x > 1 + x + \frac{x^2}{2}$
   (c) $x > 0 \implies e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$
   (d) $x > 0 \implies e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

   You may assume the basic properties of exponentials like $e^x > 1$ for $x > 0$ and $(e^x)' = e^x$.
   (Hint: Use question 1)

4. Let
   $$f(x) = \begin{cases} x^4 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$
   Prove that $f(x)$ is differentiable, $f'(x)$ is differentiable, but $f''(x)$ is not continuous at $x = 0$.

5. Let $f : [a, b] \to \mathbb{R}$ be differentiable. We know from class that if $f(x)$ has a local min/max at p in (a, b), then $f'(p) = 0$. Is the converse of this statement true? That is, if $f'(p) = 0$ with p in (a, b), does f have a local min/max at p? Prove this or give a counter-example.

6. Let $f : [a, b] \to \mathbb{R}$ be differentiable. We know from class that if $f'(x) > 0$ for all x then f is strictly increasing. Is the converse of this statement true? That is, if f is strictly increasing, is $f'(x) > 0$ for all x? Prove this or give a counter-example.