Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #10

Questions from Rudin:
Chapter 5: 1, 12

Solution: #1: We will prove that $f'(x)$ is identically zero and thus $f(x)$ is constant. To see this, we need to prove that

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = 0.$$

To this end, fix $\varepsilon > 0$ and set $\delta = \varepsilon$. Then for t such that $|x - t| < \delta = \varepsilon$, we have

$$\left| \frac{f(t) - f(x)}{t - x} - 0 \right| = \frac{|f(t) - f(x)|}{|t - x|} \leq \frac{(t - x)^2}{|t - x|} = |t - x| < \varepsilon.$$

Thus, $f'(x) = 0$ for all x and f is a constant function.

Solution: #12: We have

$$f(x) = \begin{cases} x^3 & x \geq 0 \\ -x^3 & x < 0 \end{cases}.$$

Thus, $f(x)$ is clearly differentiable for $x \neq 0$ with

$$f'(x) = \begin{cases} 3x^2 & x > 0 \\ -3x^2 & x < 0 \end{cases}.$$

At $x = 0$, we have

$$f'(0) = \lim_{x \to 0} \frac{|x|^3}{x} = \lim_{x \to 0} \frac{x^2|x|}{x} = \lim_{x \to 0} x|x| = 0$$

and thus,

$$f'(x) = \begin{cases} 3x^2 & x \geq 0 \\ -3x^2 & x < 0 \end{cases}.$$

Now $f'(x)$ is clearly differentiable for $x \neq 0$ with

$$f''(x) = \begin{cases} 6x & x > 0 \\ -6x & x < 0 \end{cases}.$$

At $x = 0$, we have

$$f''(0) = \lim_{x \to 0} \frac{3x^2 \frac{|x|}{x}}{x} = \lim_{x \to 0} 3|x| = 0$$

and thus,

$$f''(x) = \begin{cases} 6x & x \geq 0 \\ -6x & x < 0 \end{cases}.$$

Lastly, $f'''(0)$ does not exist since

$$\lim_{x \to 0} \frac{f''(x)}{x} = \lim_{x \to 0} \frac{6x \frac{|x|}{x}}{x} = \lim_{x \to 0} 6 \frac{|x|}{x}$$

does not exist as the left hand limit is -6 while the right hand limit is 6.

Additional questions:

1. Let X be a metric space and let p be a limit point of X. Prove the following basic properties of limits.

   (a) Let $f : X \to \mathbb{R}$ be the constant function defined by $f(x) = c$ for all $x \in X$. Prove $\lim_{x\to p} f(x) = c$.

   (b) Let $f, g : X \to \mathbb{R}$. If $\lim_{x\to p} f(x)$ exists and $\lim_{x\to p} g(x)$ exists, then $\lim_{x\to p} f(x) + g(x)$ exists and

   $$\lim_{x\to p} f(x) + g(x) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x).$$

   (c) Let $f, g : X \to \mathbb{R}$. If $\lim_{x\to p} f(x)$ exists and $\lim_{x\to p} g(x)$ exists, then $\lim_{x\to p} f(x) + g(x)$ exists and

   $$\lim_{x\to p} f(x) \cdot g(x) = \lim_{x\to p} f(x) \cdot \lim_{x\to p} g(x).$$

   Solution: (a) Fix $\varepsilon > 0$ and take $\delta = 1$ (any value will work). Then if $0 < d(x, p) < \delta$, we have $|f(x) - f(p)| = |c - c| = 0 < \varepsilon$.

   Solution: (b) Fix $\varepsilon > 0$. Then there exists $\delta_1 > 0$ such that if $0 < d(x, p) < \delta$, we have $|f(x) - L_f| < \varepsilon/2$ where $L_f = \lim_{x\to p} f(x)$. Likewise, there exists $\delta_2 > 0$ such that if $0 < d(x, p) < \delta$, we have $|g(x) - L_g)| < \varepsilon/2$ where $L_g = \lim_{x\to p} g(x)$. Set $\delta = \min\{\delta_1, \delta_2\}$. Then if $0 < d(x, p) < \delta$, we have $|f(x) - L_f| < \varepsilon/2$ and $|g(x) - L_g| < \varepsilon/2$. Thus

   $$|f(x) + g(x) - L_f - L_g| \leq |f(x) - L_f| + |g(x) - L_g| = \varepsilon/2 + \varepsilon/2 + \varepsilon.$$

   Hence $\lim_{x\to p} f(x) + g(x) = L_f + L_g = \lim_{x\to p} f(x) + \lim_{x\to p} g(x)$.

2. Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions and such that

   - $f(a) = g(a)$,
   - $f'(x) > g'(x)$ for all $x \in (a, b)$.

   Prove that $f(x) > g(x)$ for $x > a$.

   Solution: Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x)$ is positive for all x in (a, b). Thus for any $x > a$, by the Mean Value Theorem applied to the interval $[a, x]$, there is some $c \in (a, x)$ such that

   $$h'(c) = \frac{h(x) - h(a)}{x - a}.$$

   Since both $h'(c)$ and $x - a$ are positive, we have $h(x) > h(a)$. But $h(a) = f(a) - g(a) = 0$. Thus, $h(x) > 0$ which means exactly that $f(x) > g(x)$.

3. Consider the exponential function $y = e^x$. Prove the following inequalities:

   (a) $x > 0 \implies e^x > 1 + x$

   Solution: Applying question 1, take $f(x) = e^x$ and $g(x) = 1 + x$. Note $f(0) = g(0) = 1$. Then $f'(x) = e^x$ and $g'(x) = 1$. Since e^x is an increasing function, we have $e^x > 1$ for $x > 0$. Thus, $f'(x) > g'(x)$ for $x > 0$. Hence $f(x) > g(x)$ for $x > 0$ which means that $e^x > 1 + x$ for $x > 0$ as desired.

   (b) $x > 0 \implies e^x > 1 + x + \dfrac{x^2}{2}$

   Solution: Applying question 1, take $f(x) = e^x$ and $g(x) = 1 + x + x^2/2$. Note $f(0) = g(0) = 1$. Then $f'(x) = e^x$ and $g'(x) = 1 + x$. By the previous part, we know $e^x > 1 + x$ for $x > 0$. Thus, $f'(x) > g'(x)$ for $x > 0$. Hence $f(x) > g(x)$ for $x > 0$ which means that $e^x > 1 + x + x^2/2$ for $x > 0$ as desired.

(c) $x > 0 \implies e^x > 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$

Solution: Applying question 1, take $f(x) = e^x$ and $g(x) = 1+x+x^2/2+x^3/6$. Note $f(0) = g(0) = 1$. Then $f'(x) = e^x$ and $g'(x) = 1 + x + x^2/2$. By the previous part, we know $e^x > 1 + x + x^2/2$ for $x > 0$. Thus, $f'(x) > g'(x)$ for $x > 0$. Hence $f(x) > g(x)$ for $x > 0$ which means that $e^x > 1 + x + x^2/2 + x^3/6$ for $x > 0$ as desired.

(d) $x > 0 \implies e^x > 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$

Solution: Arguing inductively, assume that we know $e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!}$. Then taking $f(x) = e^x$ and $g(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$, note $f(0) = g(0) = 1$. Also, we have $f'(x) = e^x$ and $g'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!}$. Note $f(0) = g(0) = 1$. By assumption, $f'(x) > g'(x)$ for $x > 0$. Thus by question 1, $f(x) > g(x)$ for $x > 0$ as desired.

You may assume the basic properties of exponentials like $e^x > 1$ for $x > 0$ and $(e^x)' = e^x$.

(Hint: Use question 1)

4. Let

$$f(x) = \begin{cases} x^4 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Prove that $f(x)$ is differentiable, $f'(x)$ is differentiable, but $f''(x)$ is not continuous at $x = 0$.

Solution: Away from 0, $f(x)$ is clearly differentiable (chain rule, product rule, etc.) with $f'(x) = -x^2 \cos(1/x) + 4x^3 \sin(1/x)$. At 0, $f(x)$ is also differentiable with

$$f'(0) = \lim_{x \to 0} \frac{x^4 \sin(1/x)}{x} = x^3 \sin(1/x) = 0.$$

Thus,

$$f'(x) = \begin{cases} -x^2 \cos(1/x) + 4x^3 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Clearly, $f'(x)$ is differentiable away from 0 with $f''(x) = \sin(1/x) - 2x\cos(1/x) - 4x\sin(1/x) + 12x^2 \sin(1/x)$. At 0, $f(x)$ is also differentiable with

$$f''(0) = \lim_{x \to 0} \frac{-x^2 \cos(1/x) + 4x^3 \sin(1/x)}{x} = \lim_{x \to 0} -x\cos(1/x) + 4x^2 \sin(1/x) = 0.$$

Thus,

$$f''(x) = \begin{cases} \sin(1/x) - 2x\cos(1/x) - 4x\sin(1/x) + 12x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Lastly, $f''(x)$ is not continuous at $x = 0$ as $\lim_{x \to 0} \sin(1/x) - 2x\cos(1/x) - 4x\sin(1/x) + 12x^2 \sin(1/x)$ does not exist (since $\lim_{x \to 0} -2x\cos(1/x) - 4x\sin(1/x) + 12x^2 \sin(1/x) = 0$ and $\lim_{x \to 0} \sin(1/x)$ does not exist.

5. Let $f : [a, b] \to \mathbb{R}$ be differentiable. We know from class that if $f(x)$ has a local min/max at p in (a, b), then $f'(p) = 0$. Is the converse of this statement true? That is, if $f'(p) = 0$ with p in (a, b), does f have a local min/max at p? Prove this or give a counter-example.

Solution: This is not true. Take for example $f(x) = x^3$. Then $f'(0) = 0$ but 0 is neither a local max nor a local min as $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$.

6. Let $f : [a, b] \to \mathbb{R}$ be differentiable. We know from class that if $f'(x) > 0$ for all x then f is strictly increasing. Is the converse of this statement true? That is, if f is strictly increasing, is $f'(x) > 0$ for all x? Prove this or give a counter-example.

   Solution: This is not true. Again take $f(x) = x^3$. Then $f(x)$ is strictly increasing, but $f'(0) = 0$.