Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #1 Solutions

1. Prove that $\mathbb{Q}$ is closed under addition and multiplication. That is, show that if $x, y \in \mathbb{Q}$, then $x+y \in \mathbb{Q}$ and $xy \in \mathbb{Q}$.

   *Solution*: Since $x, y \in \mathbb{Q}$, we have $x = \frac{a}{b}$, $y = \frac{c}{d}$ with $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$. Note that

   $$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad xy = \frac{ac}{bd}$$

   Since $\mathbb{Z}$ is closed under addition and multiplication, we have that $ad + bc, ac, bd \in \mathbb{Z}$. Moreover, since $b, d \neq 0$, we have $bd \neq 0$. Thus, we have expressed each of $x + y$ and $xy$ as a ratio of two integers with the second integer non-zero. This means exactly that $x + y$ and $xy$ are in $\mathbb{Q}$ as desired.

2. What error occurs in the below "proof" to cause such a ridiculous conclusion? Let $x = y$. Then

   $$\begin{aligned} x^2 &= xy \\ x^2 - y^2 &= xy - y^2 \\ (x + y)(x - y) &= y(x - y) \\ x + y &= y \\ 2y &= y \\ 2 &= 1 \end{aligned}$$

   *Solution*: Going from the third line to the fourth line used the "fact" that $ba = ca$ implies $b = c$. However, this is not true if $a = 0$, and the $a$ in this case is $x - y$ which is indeed 0.

3. Let $\mathbb{I}$ denote the set of irrational numbers (that is the set of real numbers which are not in $\mathbb{Q}$).

   (a) Is $\mathbb{I}$ closed under addition? If so, prove it. If not, give a counter-example.

   *Solution*: No, $\mathbb{I}$ is not closed under addition. For example, we know that $\sqrt{2} \in \mathbb{I}$ and $-\sqrt{2} \in \mathbb{I}$. However, $\sqrt{2} + -\sqrt{2} = 0$ which is clearly in $\mathbb{Q}$.

   (b) Is $\mathbb{I}$ closed under multiplication? If so, prove it. If not, give a counter-example.

   *Solution*: No, $\mathbb{I}$ is not closed under multiplication. For example, we know that $\sqrt{2} \in \mathbb{I}$ and $\frac{1}{\sqrt{2}} \in \mathbb{I}$. However, $\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1$ which is clearly in $\mathbb{Q}$.

   (c) Rudin: Chapter 1, exercise 1.

   *Solution*: We are given that $r \in \mathbb{Q}$ and $x \in \mathbb{I}$, and we seek to show that $r + x \in \mathbb{I}$. Seeking a contradiction, we assume $r + x \in \mathbb{Q}$. Since $r \in \mathbb{Q}$, clearly $-r \in \mathbb{Q}$. (To see this, write $r = \frac{a}{b}$, with $a, b \in \mathbb{Z}$, and then $-r = \frac{-a}{b}$ which is in $\mathbb{Q}$. )
   Thus, since $r + x \in \mathbb{Q}$ and $-r \in \mathbb{Q}$, by question #1, we have that $r + x + -r \in \mathbb{Q}$. Hence, $x \in \mathbb{Q}$ which is a contradiction as $x \in \mathbb{I}$. Thus our assumption is false, and $r + x \in \mathbb{I}$.

4. Consider the following relation on $\mathbb{R}^2$. We say

$$(x, y) < (x', y') \text{ if } x < x' \text{ or if both } x = x' \text{ and } y < y'.$$

Under this relation, which of the following are true? Explain.

(a) $(1, 2) < (2, 3)$

*Solution*: True – the first coordinate of $(1, 2)$ is less than the first coordinate of $(2, 3)$.

(b) $(1, 2) < (3, 1)$

*Solution*: True – the first coordinate of $(1, 2)$ is less than the first coordinate of $(3, 1)$.

(c) $(1, 2) < (1, 1)$

*Solution*: False – the first coordinate of $(1, 2)$ is the same as the first coordinate of $(1, 1)$, but the second coordinate of $(1, 2)$ is greater than the second coordinate of $(1, 1)$.

(d) $(1, 2) < (1, 3)$

*Solution*: True – the first coordinate of $(1, 2)$ is the same as the first coordinate of $(1, 3)$ and the second coordinate of $(1, 2)$ is less than the second coordinate of $(1, 3)$.

Does this relation give an order on $\mathbb{R}^2$ (in the sense of Rudin, definition 1.5)?

*Solution*: This relation is an order. To see this take $(x, y)$ and $(x', y')$ in $\mathbb{R}^2$. Then since $<$ is an order on $\mathbb{R}$, we know $x < x'$ $x' < x$ or $x = x'$ and that exactly one of these statements is true. In the first case, $(x, y) < (x', y')$. In the second case, $(x', y') < (x, y)$. In the final case when $x = x'$, we then note that $y < y'$, $y' < y$ or $y = y'$. In these three additional cases, we get respectively that $(x, y) < (x', y')$, $(x', y') < (x, y)$ and $(x, y) = (x', y')$ as desired.