Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #2

1. Recall the notion of a **relation** $\mathcal{R}$ on a set S as defined in class. We further define the following concepts:

   - A relation $\mathcal{R}$ is called **reflective** if $x\mathcal{R}x$ is true for all $x \in S$.
   - A relation $\mathcal{R}$ is called **symmetric** if whenever $x\mathcal{R}y$ is true, then $y\mathcal{R}x$ is true as well.
   - A relation $\mathcal{R}$ is called **transitive** if whenever both $x\mathcal{R}y$ and $y\mathcal{R}z$ is true, then $x\mathcal{R}z$ is true.

   For each of the following relations, determine which of the above three properties it satisfies. No proofs are needed here, but if you claim that a relation does **not** satisfy one of these properties, you must give an (explicit) counter-example.

   (a) $S = \mathbb{R}$ and $\mathcal{R}$ is given by $<$

   (b) $S = \mathbb{R}$ and $\mathcal{R}$ is given by $\leq$

   (c) $S = \mathbb{Z}$ and $\mathcal{R}$ is given by $|$ (that is "divides" as in class)

   (d) Let
   $$S = \{(a, b) \in \mathbb{Z}^2 \mid b \neq 0\}$$
   and $\mathcal{R}$ is defined as follows:
   $$(a, b)\mathcal{R}(c, d) \text{ if and only if } ad = bc.$$

2. **(Definitions)** Give complete and mathematically accurate definitions of the terms *upper bound* and *least upper bound* (of a subset of $\mathbb{R}$).

3. **(True/False)** In each case, state whether the assertion is true or false. If the assertion is true then it is enough to say so. If the assertion is false then give a counterexample.

   (a) If $\theta \in \mathbb{Q}$, then $\sup\{\cos(n\theta) : n \in \mathbb{Z}\} = 1$.

   (b) The least upper bound of a nonempty subset of $\mathbb{Q}$ which is bounded above never belongs to $\mathbb{Q}$.

   (c) If a subset X of $\mathbb{R}$ has a least upper bound ℓ then $\ell \in X$.

   (d) If a subset X of $\mathbb{R}$ has a least upper bound ℓ then $\ell \notin X$.

   (e) If a subset of $\mathbb{R}$ has an upper bound then the upper bound is unique.

4. For $a, b \in \mathbb{R}$, let E denote the open interval (a, b). Prove that $\sup E = b$.

5. Rudin – Chapter 1, exercise 4
   Is this exercise still true if E is no longer assumed to be non-empty? Is it possible in this exercise for α to equal β? If so, when does this occur?

6. Rudin – Chapter 1, exercise 5

7. Rudin – Chapter 1, exercise 8

8. Prove Proposition 1.15 in Rudin.