Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #2 Solutions

1. Recall the notion of a **relation** $\mathcal{R}$ on a set S as defined in class. We further define the following concepts:

   - A relation $\mathcal{R}$ is called **reflective** if $x\mathcal{R}x$ is true for all $x \in S$.
   - A relation $\mathcal{R}$ is called **symmetric** if whenever $x\mathcal{R}y$ is true, then $y\mathcal{R}x$ is true as well.
   - A relation $\mathcal{R}$ is called **transitive** if whenever both $x\mathcal{R}y$ and $y\mathcal{R}z$ is true, then $x\mathcal{R}z$ is true.

   For each of the following relations, determine which of the above three properties it satisfies. No proofs are needed here, but if you claim that a relation does **not** satisfy one of these properties, you must give an (explicit) counter-example.

   (a) $S = \mathbb{R}$ and $\mathcal{R}$ is given by $<$

   Solution: The reflexive property fails as, for instance, $1 < 1$ is false. The symmetric property is false as $1 < 2$ holds while $2 < 1$ is false. The transitive property is true.

   (b) $S = \mathbb{R}$ and $\mathcal{R}$ is given by $\leq$

   Solution: The reflexive property holds since it is always true that $a \leq a$. The symmetric property is false as $1 \leq 2$ holds while $2 \leq 1$ is false. The transitive property is true.

   (c) $S = \mathbb{Z}$ and $\mathcal{R}$ is given by $|$ (that is "divides" as in class)

   Solution: The reflexive property holds since it is always true that $a|a$ as $a \cdot 1 = a$. The symmetric property is false as $1|2$ holds while $2|1$ is false. The transitive property is true.

   (d) Let
   $$S = \{(a, b) \in \mathbb{Z}^2 \mid b \neq 0\}$$
   and $\mathcal{R}$ is defined as follows:
   $$(a, b)\mathcal{R}(c, d) \text{ if and only if } ad = bc.$$

   Solution: All three properties hold. Reflective holds since $(a, a)\mathcal{R}(a, a)$ as $a \cdot a = a \cdot a$. Symmetric holds as
   $$(a, b)\mathcal{R}(c, d) \implies ad = bc \implies cb = da \implies (c, d)\mathcal{R}(a, b).$$
   Lastly, transitive holds since if $(a, b)\mathcal{R}(c, d)$ and $(c, d)\mathcal{R}(e, f)$, then $ad = bc$ and $cf = de$. Hence, $bcf = bde$ and thus $adf = bde$. Since $d \neq 0$, we get $af = be$ which means precisely that $(a, b)\mathcal{R}(e, f)$.

2. **(Definitions)** Give complete and mathematically accurate definitions of the terms *upper bound* and *least upper bound* (of a subset of $\mathbb{R}$).

3. **(True/False)** In each case, state whether the assertion is true or false. If the assertion is true then it is enough to say so. If the assertion is false then give a counterexample.

   (a) If $\theta \in \mathbb{Q}$, then $\sup\{\cos(n\theta) : n \in \mathbb{Z}\} = 1$.

   Solution: True. Clearly, 1 is an upper bound since $\cos(x) \leq 1$ for all x. Moreover $\cos(0 \cdot \theta) = \cos(0) = 1$ and hence no smaller bound is possible.

   (b) The least upper bound of a nonempty subset of $\mathbb{Q}$ which is bounded above never belongs to $\mathbb{Q}$.

   Solution: False. Take $E = [0, 1]$. Then $\sup(E) = 1$ which is in $\mathbb{Q}$.

(c) If a subset X of $\mathbb{R}$ has a least upper bound ℓ then $\ell \in X$.

Solution: False. Take $X = [0, 1)$. Then $\sup(E) = 1$ which is not in X.

(d) If a subset X of $\mathbb{R}$ has a least upper bound ℓ then $\ell \notin X$.

Solution: False. Take $X = [0, 1]$. Then $\sup(E) = 1$ which is in X.

(e) If a subset of $\mathbb{R}$ has an upper bound then the upper bound is unique.

Solution: False. Take $X = [0, 1)$. Then X has an upper bound of both 1 and 2 (to just mention a few).

4. For $a, b \in \mathbb{R}$, let E denote the open interval (a, b). Prove that $\sup E = b$.

   Solution: By definition, E is bounded above by b. Now let β denote some upper bound of E. Assume $\beta < b$. Then $\gamma = \frac{\beta+b}{2}$ is in E (as $\gamma < b$). However, since $\beta < \gamma$, this contradicts the fact that β is an upper bound of E. Thus, $b \leq \beta$, and we have proven that b is the least upper bound of E.

5. Rudin – Chapter 1, exercise 4
   Is this exercise still true if E is no longer assumed to be non-empty? Is it possible in this exercise for α to equal β? If so, when does this occur?

   Solution: Since E is non-empty, let $x \in E$. Then since α is a lower bound of E, we have $\alpha \leq x$. Since β is an upper bound, $x \leq \beta$. Putting these two inequalities together gives $\alpha \leq \beta$.

   If E is empty then the statement is not true. Indeed, 1 is a lower bound of the empty set (vacuously) while 0 is an upper bound of the empty set.

   In this exercise, if $\alpha = \beta$, then for any element $x \in E$, we have
   $$\alpha \leq x \leq \beta = \alpha.$$
   Thus, $\alpha = x = \beta$, and E is just a set with 1 element.

6. Rudin – Chapter 1, exercise 5

   Solution: To claim $\inf(A) = -\sup(-A)$, we need to check that $-\sup(-A)$ is a lower bound of A, and that it is the greatest lower bound. To check the first statement, we know that for any $x \in A$, we have $-x \in -A$, and thus $-x \leq \sup(-A)$. Hence, $x \geq -\sup(-A)$ as desired. To check the second statement, assume that β is some lower bound of A. Then $-\beta$ is an upper bound of $-A$. Hence, $\sup(-A) \leq -\beta$. This implies that $\beta \leq -\sup(-A)$. Thus, $-\sup(-A)$ is the greatest lower bound of A.

7. Rudin – Chapter 1, exercise 8

   Solution: Assume $\mathbb{C}$ is an ordered field. Then by Prop 1.18(d), we have $i^2 > 0$. Thus $-1 > 0$ and hence $0 > 1$. But, again by Prop 1.18(d), we know that $1 > 0$. This is a contradiction and hence $\mathbb{C}$ cannot be an order field.

8. Prove Proposition 1.15 in Rudin.

   Solution: (a) Since $x \neq 0$, $xy = xz$ implies $x^{-1}(xy) = x^{-1}(xz)$. By M3, $(x^{-1}x)y = (x^{-1}x)z$. By M5, $1y = 1z$. By M4, $y = z$.

   (b) Take $z = 1$ in (a).

   (c) Take $z = x^{-1}$ in (a).

   (d) Since $x^{-1}x = 1$, by (c) we have $x = (x^{-1})^{-1}$.