Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #3

1. Let $z = a + bi$ be a complex number which for the moment we will think of as a vector in $\mathbb{R}^2$ given by (a, b). Let $\theta(z)$ denote the angle between the positive x-axis and z satisfying $0 \leq \theta(z) < 2\pi$. (We call $\theta(z)$ the *argument* of z.) Also, let $|z| = \sqrt{a^2 + b^2}$ which we call the *length* of z.

   For $x \in \mathbb{R}$, we *define*
   $$e^{ix} = \cos(x) + i\sin(x) \in \mathbb{C}.$$

   (a) Prove that if $z \in \mathbb{C}$, then $z = re^{i\theta}$ where $r = |z|$ and $\theta = \theta(z)$.

   (b) Prove that
   $$re^{i\theta} \cdot se^{i\psi} = rse^{i(\theta+\psi)}.$$
   (In words this means that when you multiply complex numbers, you multiply the lengths and add the arguments.)

   (c) Verify that $e^{i\pi} = -1$.

2. The equation $z^n = 1$ has n solutions in $\mathbb{C}$. Write out these solutions explicitly for $n = 3, 4, 6$, and 8. Here "explicitly" means that the solutions should be written in the form $x + iy$ with x and y written in terms of rational numbers or square roots of rational numbers – no sines and cosines are allowed in your final answer.

   (Hint: The last exercise will help here.)

3. Rudin, Chapter 1: 10

4. For every $x \in \mathbb{R}$, prove that there exists a $y \in \mathbb{R}$ such that $y^3 = x$.

5. Prove that $\mathbb{C}$ satisfies the following field axioms: A4,A5,M3,M4,M5.

6. Let A and B be two sets both contained in a larger set X. We define the union $A \cup B$ to be the subset of X consisting of elements that are in A or in B. We define the intersection $A \cap B$ to be the subset of X consisting of elements that are in both A and in B. Further, we define the complement A^c to be the subset of elements of X which are not in A. Prove each of the following.

   (a) $(A \cap B)^c = A^c \cup B^c$.

   (b) $(A \cup B)^c = A^c \cap B^c$.

   (c) $(A^c)^c = A$.

7. Let S be a set and let $\sim$ be a relation on S. Assume that $\sim$ is reflexive, symmetric, and transitive. (Such a relation is called an equivalence relation.) For an element $x \in S$, we define the *equivalence class* of x, denoted by $[x]$, by
   $$[x] = \{y \in S \ : \ y \sim x \text{ holds}\}.$$
   Prove each of the following:

   (a) $x \in [x]$;

   (b) if $x \sim y$ holds, then $[x] = [y]$;

   (c) if $[x] \cap [y]$ is non-empty, then $x \sim y$ holds.

   We note for the next exercise that we use the notation $S/\sim$ to denote the collection of equivalence classes of S under $\sim$. That is, the *elements* of $S/\sim$ are *subsets* of S of the form $[x]$.

8. Let S denote the collection of ordered pairs (a,b) with $a,b \in \mathbb{Z}$ and $b \neq 0$. Recall the equivalence relation $\sim$ on the last problem set given by

$$(a,b) \sim (c,d) \text{ if and only if } ad = bc.$$

We *define* $\mathbb{Q}$ to be $S/\sim$ – that is, $\mathbb{Q}$ is the collection of equivalence classes of ordered pairs (a,b) under $\sim$.

(a) Verify that $(1,2) \sim (2,4) \sim (3,6)$. (This is meant to represent the fact that $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$.)

(b) Write down 3 elements in the equivalence class $[(5,12)]$.

(c) We *define* addition on $\mathbb{Q}$ as follows:

$$[(a,b)] + [(c,d)] = [(ad + bc, bd)].$$

(This is meant to represent that $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.)
However, it is not clear that this formula is *well-defined*. For instance, applying the formula we get $[(1,2)] + [(1,2)] = [(4,4)]$. However, $[(1,2)]$ can be written as $[(a,b)]$ for many other choices of $[(a,b)]$ (e.g. $(2,4)$). Will changing this representative of equivalence class change the value of the above addition? Let's try one example: $[(2,4)] + [(1,2)] = [(8,8)]$. Fortunately, $(4,4) \sim (8,8)$ so the resulting equivalence class is the same.
Check this now in general. Prove that

$$(a,b) \sim (a',b') \text{ and } (c,d) \sim (c',d') \text{ implies } (ad + bc, bd) \sim (a'd' + b'c', b'd').$$

(d) We *define* multiplication on $\mathbb{Q}$ as follows:

$$[(a,b)] \cdot [(c,d)] = [(ac, bd)].$$

(This is meant to represent that $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.)
Verify that this operation is well-defined. That is, prove that

$$(a,b) \sim (a',b') \text{ and } (c,d) \sim (c',d') \text{ implies } (ac, bd) \sim (a'c', b'd').$$

(e) In fact one can prove that $\mathbb{Q}$ is a field under $+$ and $\cdot$ (assuming basic properties of arithmetic on $\mathbb{Z}$). I'll ask you to do part of this: verify the following field axioms: A2,A4,A5,M4,M5.