Introduction to Analysis – MA 511 – Fall 2018 – R. Pollack
HW #3

1. Let $z = a + bi$ be a complex number which for the moment we will think of as a vector in $\mathbb{R}^2$ given by (a, b). Let $\theta(z)$ denote the angle between the positive x-axis and z satisfying $0 \leq \theta(z) < 2\pi$. (We call $\theta(z)$ the *argument* of z.) Also, let $|z| = \sqrt{a^2 + b^2}$ which we call the *length* of z.

   For $x \in \mathbb{R}$, we *define*
$$e^{ix} = \cos(x) + i\sin(x) \in \mathbb{C}.$$

   (a) Prove that if $z \in \mathbb{C}$, then $z = re^{i\theta}$ where $r = |z|$ and $\theta = \theta(z)$.

   Solution: We have
$$|z|e^{i\theta} = |z|(\cos(\theta(z)) + i\sin(\theta(z))) = |z|\cos(\theta(z)) + |z|\sin(\theta(z))i.$$
   But, by basic trigonometry, for $z \in \mathbb{C}$, we also have
$$z = |z|\cos(\theta(z)) + |z|\sin(\theta(z))i$$
   and thus $z = |z|e^{i\theta(z)}$.

   (b) Prove that
$$re^{i\theta} \cdot se^{i\psi} = rse^{i(\theta+\psi)}.$$
   (In words this means that when you multiply complex numbers, you multiply the lengths and add the arguments.)

   Solution:
$$\begin{aligned} re^{i\theta} \cdot se^{i\psi} &= rs(\cos(\theta) + i\sin(\theta))(\cos(\psi) + i\sin(\psi)) \\ &= rs((\cos(\theta)\cos(\psi) - \sin(\theta)\sin(\psi)) + i(\cos(\theta)\sin(\psi) + \cos(\psi)\sin(\theta)) \\ &= rs(\cos(\theta + \psi) + i\sin(\theta + \psi)) \\ &= rse^{i(\theta+\psi)} \end{aligned}$$

   (c) Verify that $e^{i\pi} = -1$.

   Solution: We have $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$.

2. The equation $z^n = 1$ has n solutions in $\mathbb{C}$. Write out these solutions explicitly for $n = 3, 4, 6$, and 8. Here "explicitly" means that the solutions should be written in the form $x + iy$ with x and y written in terms of rational numbers or square roots of rational numbers – no sines and cosines are allowed in your final answer.

   (Hint: The last exercise will help here.)

   Solution: Since $\left(e^{2\pi ia/n}\right)^n = 1$ for all a, we can explicitly see the n solutions to $z^n = 1$ as
$$z = 1, e^{2\pi i/n}, e^{2\pi i2/n}, \ldots, e^{2\pi ia/n}, \ldots, e^{2\pi i(n-1)/n}.$$
   For $n = 3$, the three solutions are 1,
$$e^{2\pi i/3} = \cos(2\pi/3) + \sin(2\pi/3)i = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

and

$$e^{4\pi i/3} = \cos(4\pi/3) + \sin(4\pi/3)i = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

For $n = 4$, the four solutions are 1,

$$e^{2\pi i/4} = \cos(\pi/2) + \sin(\pi/2)i = i,$$

$$e^{2\pi i 2/4} = e^{\pi i} = -1,$$

and

$$e^{2\pi i 3/4} = \cos(3\pi/2) + \sin(3\pi/2)i = -i.$$

For $n = 6$, the six solutions are the three listed for $n = 3$, together with

$$e^{2\pi i/6} = \cos(\pi/3) + \sin(\pi/3)i = \frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

$$e^{2\pi i 3/6} = -1$$

and

$$e^{2\pi i 5/6} = \cos(5\pi/3) + \sin(5\pi/3)i = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

For $n = 8$, the eight solutions are the four listed for $n = 4$, together with

$$e^{2\pi i/8} = \cos(\pi/4) + \sin(\pi/4)i = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$$

$$e^{2\pi i 3/8} = \cos(3\pi/4) + \sin(3\pi/4)i = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$$

$$e^{2\pi i 5/8} = \cos(5\pi/4) + \sin(5\pi/4)i = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,$$

and

$$e^{2\pi i 7/8} = \cos(7\pi/4) + \sin(7\pi/4)i = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$$

3. Rudin, Chapter 1: 10

4. For every $x \in \mathbb{R}$, prove that there exists a $\alpha \in \mathbb{R}$ such that $\alpha^3 = x$.

   Solution: Compare to Rudin Theorem 1.21.

5. Prove that $\mathbb{C}$ satisfies the following field axioms: A4,A5,M3,M4,M5.

   Solution: A4: The additive identity is $(0,0)$. Indeed, $(a,b) + (0,0) = (a+0, b+0) = (a,b)$ as 0 is the additive identity of $\mathbb{R}$.

   A5: For each $(a,b) \in \mathbb{C}$, note that $(a,b) + (-a,-b) = (a + -a, b + -b) = 0$ (by A5 for $\mathbb{R}$). Thus, the additive inverse $-(a,b)$ is given by $(-a,-b)$.

   M3: We have

$$\begin{aligned}((a,b)\cdot(c,d))\cdot(e,f) &= (ac-bd, ad+bc)\cdot(e,f)\\ &= ((ac-bd)e-(ad+bc)f, (ac-bd)f+(ad+bc)e)\\ &= (ace-bde-adf-bcf, acf-bdf+ade+bce)\end{aligned}$$

while

$$
\begin{aligned}
(a,b)\cdot((c,d)\cdot(e,f)) &= (a,b)\cdot(ce-df, cf+de)\\
&= (a(ce-df)-b(cf+de), a(cf+de)+b(ce-df))\\
&= (ace-adf-bcf-bde, acf+ade+bce-bdf)
\end{aligned}
$$

Comparing these two expressions we get

$$((a,b)\cdot(c,d))\cdot(e,f) = (a,b)\cdot((c,d)\cdot(e,f))$$

as desired. Note that A3 and M3 for $\mathbb{R}$ are being used implicitly here as we have not included the proper parentheses everywhere.

M4: The multiplicative identity is $(1,0)$. Indeed, $(a,b)\cdot(1,0) = (a\cdot 1 - b\cdot 0, a\cdot 0 + b\cdot 1) = (a,b)$. Here we are using M4 for $\mathbb{R}$ along with $0x = 0$ for all $x \in \mathbb{R}$.

M5: For each non-zero $(a,b) \in \mathbb{C}$, note that

$$(a,b)\cdot\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = \left(\frac{a^2+b^2}{a^2+b^2}, \frac{a(-b)+ba}{a^2+b^2}\right) = (1,0).$$

Thus, the multiplicative inverse $(a,b)^{-1}$ is given by $\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$. Note that we are using that (a,b) is non-zero to know that a^2+b^2 is non-zero.

6. Let A and B be two sets both contained in a larger set X. We define the union $A \cup B$ to be the subset of X consisting of elements that are in A or in B. We define the intersection $A \cap B$ to be the subset of X consisting of elements that are in both A and in B. Further, we define the complement A^c to be the subset of elements of X which are not in A. Prove each of the following.

   (a) $(A \cap B)^c = A^c \cup B^c$.

   Solution: In each of these exercises we will check that $x \in X$ is in the left hand side if and only if it is in the right hand side. This will prove the equality of the two sets.
   We have $x \in (A \cap B)^c$ if and only if x is not in $A \cap B$ which is true if and only if x is not in A or x is not in B which is true if and only if $x \in A^c$ or $x \in B^c$ which is true if and only if x is in $A^c \cup B^c$.

   (b) $(A \cup B)^c = A^c \cap B^c$.

   Solution: We have $x \in (A \cup B)^c$ if and only if x is not in $A \cup B$ which is true if and only if x is not in A and x is not in B which is true if and only if $x \in A^c$ and $x \in B^c$ which is true if and only if x is in $A^c \cap B^c$.

   (c) $(A^c)^c = A$.

   Solution: We have $x \in (A^c)^c$ if and only if x is not in A^c which is true if and only if $x \in A$.

7. Let S be a set and let $\sim$ be a relation on S. Assume that $\sim$ is reflexive, symmetric, and transitive. (Such a relation is called an equivalence relation.) For an element $x \in S$, we define the *equivalence class* of x, denoted by $[x]$, by

$$[x] = \{y \in S \ : \ y \sim x \text{ holds}\}.$$

   Prove each of the following:

   (a) $x \in [x]$;

   Solution: Clear since $x \sim x$.

(b) if $x \sim y$ holds, then $[x] = [y]$;

Solution: Let $z \in [x]$. This means that $z \sim x$. Since $x \sim y$, by transitivity we get $z \sim y$ and $z \in [y]$. Thus, $[x] \subseteq [y]$. To prove the reverse inclusion, take $z \in [y]$. Thus, $z \sim y$. Since $x \sim y$, by symmetry, $y \sim x$, and then by transitivity we get $z \sim x$. Hence, $z \in [x]$ and we have shown $[y] \subseteq [x]$. Therefore, $[x] = [y]$.

(c) if $[x] \cap [y]$ is non-empty, then $x \sim y$ holds.

Solution: Let z be in $[x] \cap [y]$. Thus $z \sim x$ and $z \sim y$. By symmetry, we have $x \sim z$ and by transitivity, we have $x \sim y$.

We note for the next exercise that we use the notation $S/\sim$ to denote the collection of equivalence classes of S under $\sim$. That is, the *elements* of $S/\sim$ are *subsets* of S of the form $[x]$.

8. Let S denote the collection of ordered pairs (a, b) with $a, b \in \mathbb{Z}$ and $b \neq 0$. Recall the equivalence relation $\sim$ on the last problem set given by

$$(a, b) \sim (c, d) \text{ if and only if } ad = bc.$$

We *define* $\mathbb{Q}$ to be $S/\sim$ – that is, $\mathbb{Q}$ is the collection of equivalence classes of ordered pairs (a, b) under $\sim$.

(a) Verify that $(1, 2) \sim (2, 4) \sim (3, 6)$. (This is meant to represent the fact that $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$.)

Solution: Clear. For instance, $2 \cdot 2 = 1 \cdot 4$.

(b) Write down 3 elements in the equivalence class $[(5, 12)]$.

Solution: (5,12),(-5,-12),(10,24)

(c) We *define* addition on $\mathbb{Q}$ as follows:

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)].$$

(This is meant to represent that $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.)
However, it is not clear that this formula is *well-defined*. For instance, applying the formula we get $[(1, 2)] + [(1, 2)] = [(4, 4)]$. However, $[(1, 2)]$ can be written as $[(a, b)]$ for many other choices of $[(a, b)]$ (e.g. $(2, 4)$). Will changing this representative of equivalence class change the value of the above addition? Let's try one example: $[(2, 4)] + [(1, 2)] = [(8, 8)]$. Fortunately, $(4, 4) \sim (8, 8)$ so the resulting equivalence class is the same.
Check this now in general. Prove that

$$(a, b) \sim (a', b') \text{ and } (c, d) \sim (c', d') \text{ implies } (ad + bc, bd) \sim (a'd' + b'c', b'd').$$

Solution: We are given that $ab' = a'b$ and $cd' = c'd$. We compute

$$(ad+bc)(b'd')-(bd)(a'd'+b'c') = ab'dd'+bb'cd'-a'bdd'-bb'c'd = a'bdd'+bb'c'd'-a'bdd'-bb'c'd = 0.$$

Thus, $(ad + bc, bd) \sim (a'd' + b'c', b'd')$.

(d) We *define* multiplication on $\mathbb{Q}$ as follows:

$$[(a, b)] \cdot [(c, d)] = [(ac, bd)].$$

(This is meant to represent that $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.)

Verify that this operation is well-defined. That is, prove that

$$(a, b) \sim (a', b') \text{ and } (c, d) \sim (c', d') \text{ implies } (ac, bd) \sim (a'c', b'd').$$

Solution: We are given that $ab' = a'b$ and $cd' = c'd$. We compute

$$acb'd' - a'c'bd = a'bcd' - a'bcd' = 0$$

and $(ac, bd) \sim (a'c', b'd')$.

(e) In fact one can prove that $\mathbb{Q}$ is a field under $+$ and $\cdot$ (assuming basic properties of arithmetic on $\mathbb{Z}$). I'll ask you to do part of this: verify the following field axioms: A2,A4,A5,M4,M5.

Solution:
A2: We have

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)] = [(da + cb, db)] = [(cb + da, db)] = [(c, d)] + [(a, b)].$$

Here we are using A2 and M2 for $\mathbb{Z}$.

A4: The additive identity is $[(0, 1)]$ since

$$[(a, b)] + [(0, 1)] = [(a \cdot 1 + b \cdot 0, b \cdot 1)] = [(a, b)].$$

Here we are using A4 and M4 for $\mathbb{Z}$.

A5: For each $[(a, b)] \in \mathbb{Q}$, note that

$$[(a, b)] + [(-a, b)] = [(ab + b(-a), b^2)] = [(0, b^2)] = [(0, 1)]$$

as $(0, b^2) \sim (0, 1)$. Thus, the additive inverse $-[(a, b)]$ is given by $[(-a, b)]$.

M4: The multiplicative identity is $[(1, 1)]$ since

$$[(a, b)] \cdot [(1, 1)] = [(a \cdot 1, b \cdot 1)] = [(a, b)].$$

Here we are using M4 for $\mathbb{Z}$.
A5: For each non-zero $[(a, b)] \in \mathbb{Q}$, note that

$$[(a, b)] \cdot [(b, a)] = [(ab, ab)] = [(1, 1)]$$

as $(ab, ab) \sim (1, 1)$. Thus, the multiplicative inverse $[(a, b)]^{-1}$ is given by $[(b, a)]$. Further note that this makes sense as $a \neq 0$.